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Ferromagnetic Superconductors

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The properties of ferromagnetic superconductors are considered. It is shown that the presence of spontaneous magnetization complicates the detection of ferromagnetic superconductors. However, use of thin films or wires, under known conditions and with large samples of high coercive force, should materially increase the possibility of detection of superconductivity in ferromagnetic metals and alloys.

TO date, superconductivity has never been detected in a ferromagnetic metal or alloy. This circumstance would not ordinarily be surprising, since many nonferromagnetic metals also fail to exhibit superconductivity.

On the other hand, in the present state of the microscopic theory of superconductivity, there exists no basis for denying the possibiliity of the existence of superconducting ferromagnets. Furthermore, in favor of such a possibility stands the fact that superconductivity is fundamentally related to the outermost electrons, while ferromagnetism is connected with the more deeply seated electrons. Moreover, many superconductors and ferromagnets are distinguished, at least insofar as their atomic states are concerned, only by d- or even f-electrons. Thus, superconducting lanthanum and hafnium differ from ferromagnetic gadolinium only in the $4f$- and $5d$-electrons,* while the ferromagnetic Fe, Ni and Co differ from superconducting Ti and V only by the addition of several $3d$-electrons. Thus the absence of superconducting ferromagnets is, in each case, not a self-evident factor and requires an explanation.

The purpose of the present research is to point out the almost complete impossibility in practice, under ordinary conditions, of observing superconductivity in any sort of ferromagnets. It is shown that the presence in ferromagnets of spontaneous magnetization M_0 causes a large sample, even in the absence of an external magnetic field, to possess an induction $B_0 = 4\pi M_0$. Furthermore, it is quite natural, as will be shown below, that, in equilibrium, the superconducting phases can exist in large ferromagnets only if $B_0 = 4\pi M_0 < H^0_{c1}(0)$, where $H^0_{c1}(0)$ is the critical field of the large sample at $T = 0$ and consideration is not given to the effect of magnetization. The value of B_0 for $T = 0$ for Fe, Co, Ni, and Gd is, respectively, 22,000, 18,500, 6400 and 24,800 G, while $H^0_{c1}(0)$ for different superconducting elements lies in the range from 2600 G for Nb to 28.8 G for Cd.

Thus the probability of finding the superconductivity of ferromagnets in ordinary measurements is as small as in the case of nonferromagnetic superconductors placed in an external field with a magnetization of several thousand oersteds. In other words, a large ferromagnet can under ordinary conditions undergo a transition to the superconducting state only when its spontaneous magnetization M_0 (for $T=0$) is very small ($B_0 = 4\pi M_0 \lesssim 10^3$), and the critical field is very large ($\gtrsim 10^3$).

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*It is possible that there are even closer superconducting neighbors to gadolinium since the question of the superconductivity of the rare earths is unfortunately not completely resolved. Moreover, there are several other ferromagnetic rare earths besides gadolinium.[1]

These observations do not mean, however, that superconductivity of ferromagnets for which $B_0 = 4\pi M_0 > H^0_{c1}(0)$ can never be observed. On the contrary, under definite conditions (for thin films and wires, and also, perhaps, for large samples with high coercive force) superconductivity of ferromagnets can be observed if such superconductivity is otherwise possible (i.e., in the absence of the above-mentioned masking magnetic effect associated with the presence of spontaneous magnetization M_0).

1. In weak magnetic fields, $H \ll H_{cr}$, where H_{cr} is the critical field, the behavior of nonmagnetic superconductors is well described by the equations[*]

$$\operatorname{curl} \Lambda \mathbf{j}_s = -(1/c)\,\mathbf{H}, \qquad (1)$$

$$\partial \Lambda \mathbf{j}_s / \partial t = \mathbf{E}. \qquad (2)$$

Furthermore, in the static case, to which we limit ourselves below, we must consider the field equation

$$\operatorname{curl} \mathbf{H} = (4\pi/c)\,\mathbf{j}_s. \qquad (3)$$

We get from (1) and (3) in the usual way (we assume that $\Lambda = \text{const}$):

$$\Delta \mathbf{H} - \delta_0^{-2}\mathbf{H} = 0, \quad \Delta \mathbf{j}_s - \delta_0^{-2}\mathbf{j}_s = 0, \qquad (4)$$

$$\delta_0^2 = c^2\Lambda/4\pi = mc^2/4\pi e^2 n_s.$$

Above, no distinction has been made between the field H and the induction $B = H + 4\pi M$. For para- and diamagnets, this is practically true, but it is otherwise in the case of ferromagnets. It is easy to see that for $B \neq H$, Eq. (1) can be replaced by

$$\operatorname{curl} \Lambda \mathbf{j}_s = -(1/c)\,\mathbf{B}. \qquad (5)$$

This substitution can be based on a whole series of considerations, namely: Eq. (5) and not (1) is compatible in the general case with the field equations and with Eq. (2). The latter follows from the very existence of superconductivity and is therefore preserved even in the transition to the case of magnets[*]; evidence of this is given both by quantum mechanical considerations and also by the simple fact that the induction B is the field which acts on the current and which also determines the density of the superconducting current.

We get from (3) and (5) that

$$\Delta \mathbf{B} - \delta_0^{-2}\mathbf{B} = -4\pi \operatorname{curl}\operatorname{curl} \mathbf{M}, \qquad (6)$$
$$\Delta \mathbf{j}_s - \delta_0^{-2}\mathbf{j}_s = c\,\delta_0^{-2}\operatorname{curl} \mathbf{M}.$$

We shall consider below that the ferromagnetic is "ideal", i.e.,

$$\mathbf{B} = \mu\mathbf{H} + 4\pi\mathbf{M}_0, \quad \mathbf{M} = \mathbf{M}_0 + \frac{\mu - 1}{4\pi}\mathbf{H}, \qquad (7)$$

where M_0 is the spontaneous magnetization and μ does not depend on H. For liquid helium temperatures, we can assume that M_0 is independent of temperature; at saturation, which is the usual condition, we can also set $\mu = 1$. Making use of (7), we have (it is assumed that $\mu = \text{const}$):

$$\Delta \mathbf{B} - \delta^{-2}\mathbf{B} = -4\pi \operatorname{curl}\operatorname{curl} \mathbf{M}_0,$$
$$\Delta \mathbf{j}_s - \delta^{-2}\mathbf{j}_s = (c/\mu\delta^2)\operatorname{curl} \mathbf{M}_0,$$
$$\delta^2 = \delta_0^2/\mu. \qquad (8)$$

It is understood that these formulas are suitable for para- and diamagnetic materials for $M_0 = 0$. It is clear from (8) that in regions where $M_0 = \text{const}$, the induction and the superconducting current are damped out within the penetration depth δ. On the boundaries of the domains, where $\operatorname{curl} M_0 \neq 0$, there are sources of induction and of current. If the sample consists of a single domain (which is also assumed below if there is no

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[*] We avoid here the problem of the validity of Eqs. (1) and (2), inasmuch as, in our opinion, there is no substantial reason for doubting them at the present time.[2]

[*] It follows from (2) and from the equation $\operatorname{curl} E = -(1c)\,\partial B/\partial t$ that

$$\frac{\partial}{\partial t}\left\{\operatorname{curl}\Lambda \mathbf{j}_s + \frac{1}{c}\mathbf{B}\right\} = 0,$$

which is compatible with (5)

contrary limitation), then it is possible to set curl $\mathbf{M}_0 = 0$ everywhere. On the boundary of the specimen, where $\mathbf{M}_0$ has a discontinuity, we must use the boundary conditions

$$H_e = H_i, \ B_e = (B_i - 4\pi M_0) / \mu; \qquad (9)$$

the indices e and i refer respectively to the region outside the metal, where $\mu = 1$ and $\mathbf{M}_0 = 0$, and to the region inside the metal at its boundary. In the general case, the tangential components of H, B and $\mathbf{M}_0$ would seem to enter into Eq. (9), but this is not reflected in the results, since below, for concrete applications, the normal components will be considered equal to zero (we consider below only cylindrical samples that are parallel to the field).

It is clear from the foregoing that in the superconducting state in the absence of an external field ($H_e = 0$), $B = 0$ in the mass of a uniformly magnetized superconducting ferromagnet, while in the normal state in such a cylindrical sample, $H_i = 0$, $B_i = B_0 = 4\pi M_0$. The induction B_0 is created by the (molecular) surface current $i_{\mathrm{mol}} = cM_0$, which flows over the surface of the sample. In the superconducting state, even in the absence of an external field near the surface of the metal, the superconducting current j_s flows in a layer of thickness of the order δ. This current screens the field of the current i_{mol} in the mass of the metal. In this case the equivalent superconducting surface current $i_s = \int_0^\infty j_s\, dz$ (the z axis is directed along the normal to the surface of the sample, which is located at $z = 0$) is determined from the condition (the quantities with index ∞ refer to the mass of the sample):

$$\begin{aligned} i_s &= c\,(H_{i\infty} - H_e) / 4\pi \\ &= -cM_0 / \mu = -cH_e / 4\pi, \end{aligned}$$

since $H_{i\,\infty} = -4\pi M_0/\mu$, inasmuch as $B_{i\,\infty} = \mu H_{i\,\infty} + 4\pi M_0 = 0$. For $H_e = 0$ and $\mu = 1$, $i_s = -i_{\mathrm{mol}}$, i.e., the superconducting current accurately compensates the "molecular" current.

2. Let us go on to the problem of the destruction of superconductivity of large ferromagnets in a magnetic field (we call that sample "large" whose smallest dimensions are significantly larger than $\delta \sim 10^{-5}$ cm).

Here, as in the case of nonferromagnets, we must use only the fact that in the superconductor, $B = 0$ (we neglect the surface layer where $B \neq 0$). Moreover, it is necessary to make use of the expression for the density of magnetic energy of the ferromagnets:

$$w_m = \frac{1}{4\pi} \int_0^B \mathbf{H} \cdot d\mathbf{B} = \frac{\mu H^2}{8\pi} - \frac{2\pi M_0^2}{\mu} = \frac{B^2}{8\pi\mu} - \frac{\mathbf{M}_0 \mathbf{B}}{\mu}, \qquad (10)$$

where Eq. (7) is assumed and the constant is so chosen that $w_m = 0$ for $B = 0$.

Considering a cylindrical sample with a cross section of arbitrary form, located in an external magnetic field H_0 parallel to its axis, we obtain the following expression for the free energy per unit volume of sample:

$$\begin{aligned} F_n &= F_{n0} + \frac{\mu H_e^2}{8\pi} - \frac{2\pi M_0^2}{\mu} - MH_e \qquad (11) \\ &= F_{n0} + \frac{2-\mu}{8\pi} H_e^2 - \frac{2\pi M_0^2}{\mu} - M_0 H_e, \end{aligned}$$

where F_{n0} is the free energy after deduction of the energy of the magnetic field and the term $-MH_e = -M_0 H_e - [(\mu-1)/4\pi] H_e^2$ is the density of the energy of interaction of the sample with the external field.*

In the superconducting state the free energy will be the same as for nonferromagnets since in both cases

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*In the general case, the free energy (more accurately the corresponding thermodynamic potential) of a system in an external uniform field $\mathbf{H}_e$ has the form:

$$\int F\, dV = \int (F_0 + w_m)\, dV - \frac{1}{4\pi} \int (\mathbf{B} - \mathbf{H}_e)\, \mathbf{H}_e\, dV,$$

where the integral is carried out over all space and F_0 is the free energy for $B=0$. For the cylinder we must limit ourselves to integration over the volume of the specimen, after which we get for the energy density, referred to unit volume:

$$F = F_0 + (1/S) \int w_m dV - (H_e / 4\pi S) \int (B - H_e)\, dV, \quad (12)$$

where S is the cross section of the specimen and the integration is carried out over a section of unit length along the axis of the cylinder. In the normal state, when the field is uniform,

$$-\frac{H_e}{4\pi S} \int (B - H_e)\, dV = -\frac{H_e (B - H_e)}{4\pi} = -MH_e;$$

the latter expression has a simple meaning, well known from magnetostatics.

$$B = 0 \text{ and } - H_e (B - H_e) / 4\pi = H_e^2 / 4\pi.$$

Thus,

$$F_s = F_{s0} + H_e^2 / 4\pi. \tag{13}$$

The critical field for a large sample is determined from the equality $F_n = F_s$, whence

$$H_{c1}(T) = \pm H_{c1}^0 (T) / \sqrt{\mu} - 4\pi M_0 / \mu, \tag{14}$$

$$H_{c1}^0 = \sqrt{8\pi (F_{n0} - F_{s0})}. \tag{15}$$

The field $H_{c1}^{(0)}$ is evidently the critical magnetic field in the case of a nonferromagnetic metal with the same difference $F_{n0} - F_{s0}$ as for the ferromagnet under consideration and with $\mu = 1$. The direction of the external field H_e will be considered positive always, by virtue of which $H_{c1} \geq 0$. As concerns the magnetization M_0, in the isotropic case considered, it can be directed either along the field ($M_0 > 0$) or against the field ($M_0 < 0$).

In the first case, which is in equilibrium, for $M_0 = |M_0| > 0$ we have

$$H_{c1} (T) = H_{c1}^{(0)} (T) / \sqrt{\mu} - 4\pi M_0 / \mu \tag{16}$$

and superconductivity is possible only if

$$H_{c1}^{(0)} (0) > 4\pi M_0 / \sqrt{\mu}. \tag{17}$$

The result (16) means that the critical field $H_{c1}^{(0)} \sqrt{\mu}$ for $M_0 = 0$ must be equal to $H_{c1}^{(0)} - H_i$ where $H_i = -4\pi M_0 / \mu$ is the field for $B = 0$. For $\mu = 1$, it can also be shown that $H_{c1}^{(0)} = H_{c1} + B_0$ where $B_0 = 4\pi M_0$ is the induction in the absence of superconductivity. For a saturated specimen, $\mu = 1$ effectively, and the criterion goes over into the form originally introduced. As has already been shown, superconductivity can be observed only under extraordinary circumstances when the quantity $B_0 = 4\pi M_0$ is anomalously small*, and the field $H_{c1}^{(0)}$ (0) corresponding is sufficiently large, although it lies inside reasonable limits. A state diagram of case (16) is shown schematically in Fig. 1.

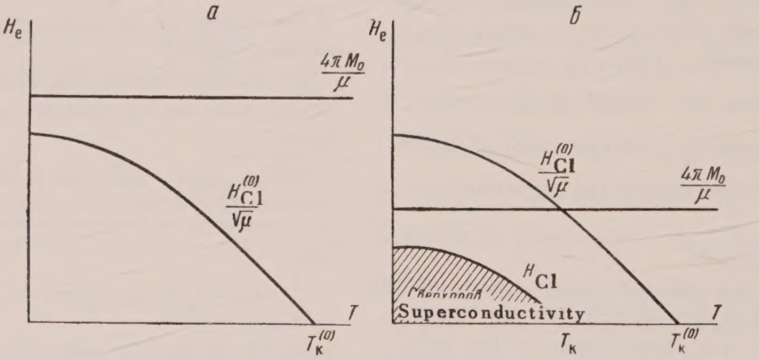

FIG. 1. $M_0 > 0$, $H_{C1}^{(0)} (0) / \sqrt{\mu} < 4\pi M_0 / \mu$, superconductivity is not possible; *(b)* $M_0 > 0$, $H_{C1}^{(0)} / \sqrt{\mu} > 4\pi M_0 / \mu$, $H_{C1} (7) = H_{C1}^{(0)} (T) / \sqrt{\mu} - 4\pi M_0 / \mu$

For sufficiently clean and perfect materials, saturation takes place at very weak fields and the calculation just run through is valid (especially if anisotropy is not taken into account or if one does not consider the field directed along the axis of easy magnetization). In the absence of the field and for a multi-domain configuration, the criterion (17) remains suitable, inasmuch as the dimensions of the domains usually far exceed

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*A sufficiently small value of M_0 can exist for alloys with concentration of components close to those for which ferromagnetism completely disappears (we assume that the transition from ferromagnetic alloys to nonferromagnetic is not a first order transition).

$\delta \sim 10^{-5}$ cm, and there must then be a current flow over the surface of the domain, canceling out the induction B in the interior of the domain. Moreover, upon appearance of superconductivity, the entire domain structure can, and in equilibrium must, change in the direction of enlargement or elimination of the domains, not to mention the fact that for long cylindrical samples, the role of the domain structure is generally relatively small.

We also note that for samples with large demagnetizing factor (for example, magnetization perpendicular to the plane of a thin disk) the field B in a sample in the nonsuperconducting state can be very small in comparison with $4\pi M_0$. Therefore the appearance of superconductivity will perhaps be facilitated in this case. However, it is also possible that edge effects will put at naught the advantage connected with reduced B. Thus the problem of the expediency of using samples with a large demagnetization factor demands special analysis which is yet to be carried out.

Let us now consider a uniformly magnetized cylindrical sample in which the magnetization $M_0 < 0$, i.e., it is directed against the field H_e. Such a state is metastable, but it is destroyed only by the achievement of a field H_c of value equal to the coercive force H_c which in some cases is of the order $4\pi M_0$, although it usually is significantly smaller than this value. In the case considered, there are, in accordance with Eq. (14), two critical field values:

$$H_{c1}^{+}(T) = \frac{4\pi |M_0|}{\mu} + \frac{H_{c1}^{(0)}(T)}{\sqrt{\mu}}, \tag{18}$$

$$H_{c1}^{-}(T) = \frac{4\pi |M_0|}{\mu} - \frac{H_{c1}^{(0)}(T)}{\sqrt{\mu}}.$$

It is easy to become convinced that the superconducting phase can exist only in a region of field H_e:

$$H_{c1}^{-}(T) \leqslant H_e \leqslant H_{c1}^{+}(T).^{*} \tag{19}$$

Here, if $H_{c1}^{-}(T)$ is negative in accord with Eq. (18), then this means that the critical field H_{c1}^{-} does not exist, and superconductivity will be observed in a zero field; a similar possibility exists upon fulfillment of condition (17). The region of existence of the superconducting phase in the case $M_0 < 0$ is clear from the schematic Fig. 2.

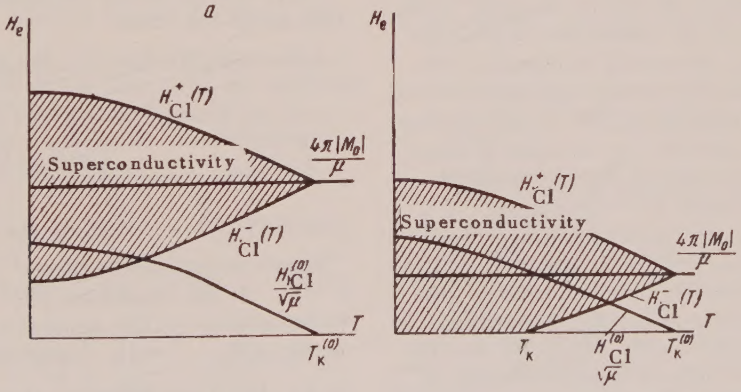

FIG. 2. *(a)* $M_0 < 0$, $H_{C1}^{(0)}/\sqrt{\mu} < 4\pi/M_0\ 1/\mu, \lesssim H_{C1}^{+}(T) = 4\pi|M_0|/\mu + H_{C1}^{(0)}(T)/\sqrt{\mu}$, $H_{C1}^{-} = 4\pi|M_0|\mu - H_{C1}^{(0)}(5)/\sqrt{\mu}$; *(b)* $M_0 < 0$, $H_{C1}^{(0)}(0)/\sqrt{\mu} < 4\pi|M_0|/\mu$, $H_{C1}^{(H)}(T) = 4\pi|M_0|/\mu + H_{C1}^{(0)}(+)/\sqrt{\mu}$,

$$H_{C1}^{-}(T) = 4\pi|M_0|/\mu - H_{C1}^{(0)}(T)/\sqrt{\mu} \gtreqless 0.$$

It is obvious that this drawing can fully correspond to the actual case only if $H_{c1}^{+}(T) < H_c$. If the coercive force $H_c < H_{c1}^{-}(0)$ then superconductivity cannot in general be observed for $M_0 < 0$; if $H_{c1}^{-}(T) < H_c$ then upon increase in the field there ought to occur a transition to the superconducting state. The question as to in what region this state will exist in the general case is difficult to answer with certainty inasmuch as we are dealing with a nonequilibrium system. From each case discussed it is clear that for use of a configuration with $M_0 < 0$, superconductivity is possible for the condition

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*We avoid the cases of supercooling or superheating, i.e., in region (19) the free energy of the superconducting phase F_s is less than or equal to the free energy of the normal phase F_n.

$$H^{(0)}_{c1}(0) > (4\pi (M_0)/\sqrt{\mu}) - \sqrt{\mu} H_c, \quad (20)$$

which is weaker than (17).

In spite of this latter circumstance, the known values of M_0 and H_c for ferromagnetic elements do not permit us to hope for the discovery of superconductivity of large samples of them. In the case of certain alloys, utilization of configurations with $M_0 < 0$ can significantly increase the chances of discovery of superconductivity. However, greatest interest attaches to the use of ferromagnetic thin films (or wires), since in this case the critical field H^0_{cr} increases strongly in comparison with H^0_{c1}, which allows us to hope for a substantial change in the situation. As will be shown, this hope is entirely justified.

3. Equation (1) is applicable only for a weak field $H << H_{cr}$ and therefore relates to Eq. (5) used in Sec. 1. On the other hand, in ferromagnetic superconductors, in contrast to nonferromagnetic ones, the field, generally speaking, is never weak everywhere. Therefore, in consideration of large samples, it is already necessary to extend to the ferromagnetic case the phenomenological theory of superconductivity developed in Ref. 3 and applied to arbitrary fields. However, in the case of large samples and in the absence of the intermediate state (and if the parameter $\varkappa << 1$; see Ref. 3), the theory of Ref. 3 leads to comparatively small departures from the theory of the Londons which is based on Eq. (1). From the point of view of Sec. 2, the chief consequence of Eq. (5) is that $B=0$ in the interior of the specimen. Inasmuch as this same conclusion also follows from the theory of Ref. 3 or, more precisely, from its generalization to ferromagnets carried out below, the results of Sec. 2 remain completely valid and the limited applicability of Eq. (5) is not important for us. In the transition to the problem of the destruction of superconductivity of specimens of small dimensions, the situation is completely different and one can start out only from the scheme put forth in Ref. 3.

In ferromagnetics, the fundamental equation for the free energy of the superconductor in the field (in the theory of Ref. 3), has, as is easy to see, the form

$$F_{sH} = F_{s0} + \frac{1}{2m}\left| - i\hbar\nabla\Psi - \frac{e}{c}\mathbf{A}\Psi\right|^2 + \frac{B^2}{8\pi\mu} - \frac{\mathbf{M}_0\mathbf{B}}{\mu}, \quad (21)$$

$$F_{s0} = F_{n0} + \alpha|\Psi|^2 + \frac{1}{2}\beta|\Psi|^4, \quad \mathbf{B} = \mathrm{curl}\,\mathbf{A}.$$

The entire difference of Eq. (21) from the corresponding equation in Ref. 3 consists of the use of Eq. (10) for the density of the magnetic energy, instead of the quantity $H^2/8\pi$ which is suitable for nonferromagnets; moreover, here curl A=B and not curl A=H. By virtue of the above, it is evident that for $M_0 = 0$ and $\mu=1$, Eq. (21) transforms into the expression for F_{sH} from Ref. 3. The equation for Ψ obtained from Eq. (21) has the same form as that in Ref. 3. Variation of Eq. (21) with respect to A under the condition div A=0 leads to the expression

$$\Delta\mathbf{A} = -\frac{4\pi}{c}(\mu\mathbf{j}_s + c\,\mathrm{curl}\,\mathbf{M}_0),$$

$$\mathbf{j}_s = (-ie\hbar/2m)(\Psi^*\nabla\Psi - \Psi^*\nabla\Psi) - (e^2/mc)\Psi^*\Psi\mathbf{A}. \quad (22)$$

This expression for A is in agreement with Eqs. (3) and (7) as it should be.

If Ψ =const, as we can assume in films[3,4], Eq. (22) takes the form:

$$\Delta\mathbf{A} - |\Psi_0|^2\delta^{-2}\mathbf{A} = -4\pi\,\mathrm{curl}\,\mathbf{M}_0,$$

$$\delta^2 = \delta_0^2/\mu = mc^2/4\pi e^2\mu|\Psi_\infty|^2,$$

$$|\Psi|^2 = |\Psi_0|^2|\Psi_\infty|^2, \quad (23)$$

where $|\Psi_\infty|^2 = -\alpha/\beta$ is the concentration of "superconducting electrons" (see Ref. 3) for $B = 0$. At the same time $|\Psi|^2$ is the analogous concentration in the general case (for $B=0$, evidently, $|\Psi_0|^2 = 1$). Equation (23) is equivalent to Eq. (8) if we consider that $n_s = |\Psi|^2 = |\Psi_0|^2|\Psi_\alpha|^2$ and the values of δ in Eq. (8) and (23) coincide completely only for $|\Psi_0|^2 = 1$; inasmuch as for a large sample the latter equality does not take place in practice, we have not introduced different designations for the quantities δ in (8) and (23).

Let us now consider a film of thickness 2d, which is parallel to a field H_e. Taking the normal to the film along the z axis (at the center of the film, $z=0$), and the direction of the field H_e along the y axis, we see that the potential A and the current density j_s have components only along the x axis.

In this case, for $A_x = A$, we get

$$(d^2A/dz^2) - (\Psi_0^2/\delta^2)A = 0,$$
$$B_y = B = dA/dz = \mu H + 4\pi M_0, \tag{24}$$

where the film is considered uniformly magnetized (therefore curl $M_0 = 0$; the field H and the magnetization M_0 are assumed also to be parallel to the y axis (like H_e).

It follows from Eq. (24) and the boundary conditions (9) that in the film

$$A = \frac{\delta B_i \operatorname{sh}(\Psi_0 z/\delta)}{\Psi_0 \operatorname{ch}(\Psi_0 d/\delta)}, \tag{25}$$
$$B = \frac{B_i \operatorname{ch}(\Psi_0 z/\delta)}{\operatorname{ch}(\Psi_0 d/\delta)},$$
$$B_i = \mu H_e + 4\pi M_0 .$$

In the absence of an external field, $B_1 = B_0 = 4\pi M_0 \neq 0$, in contrast to nonferromagnets, in which, for $H_e = 0$, $B = 0$, $A = 0$ and $j_s = 0$. Therefore, in the latter case, for $H_e = 0$, the quantity $|\Psi_0|^2 = 1$. In ferromagnets, this is no longer so. The quantity Ψ_0 in the film, both for $H_e = 0$ and for $H_e \neq 0$, is determined from the requirement of a minimum of the free energy of the film, in which, for $H_e \neq 0$, one ought also to take into account the interaction of the magnetic moment of the film with the external field. In other words, one must minimize the free energy (12), i.e., in the case under consideration of a film with Ψ=const, the expression [see Ref. (21)]:

$$F_s = \frac{1}{2d}\int_{-d}^{d}\left\{F_{sH} - H_e\left(\frac{B - H_e}{4\pi}\right)\right\}dz, \tag{26}$$
$$F_{sH} = F_{s0} + \frac{B^2}{8\pi\mu} - \frac{M_0 B}{\mu}.$$

Considering Eq. (25) and the fact that $H_{cl}^{(0)} = \sqrt{4\pi\alpha^2/\beta}$ and $|\Psi|^2 = -(\alpha/\beta)|\Psi_0|^2$, we get*

$$F_s = F_{n0} + \frac{H_e^2}{4\pi} - \frac{B_i^2}{8\pi\mu}\frac{\operatorname{th}(\Psi_0 d/\delta)}{\Psi_0 d/\delta} + \frac{(H_{cl}^{(0)})^2}{8\pi}\Psi_0^2(\Psi_0^2 - 2)$$
$$= F_n + \frac{B_i^2}{8\pi\mu}\left\{1 - \frac{\operatorname{th}(\Psi_0 d/\delta)}{\Psi_0 d/\delta}\right\} + \frac{(H_{cl}^{(0)})^2}{8\pi}\Psi_0^2(\Psi_0^2 - 2). \tag{27}$$

Here, the total free energy in the normal state (referred to unit volume), which is not different from Eq. (11), is equal to

$$F_n = F_{n0} + \frac{(2-\mu)}{8\pi}H_e^2 - \frac{2\pi M_0^2}{\mu} - M_0 H_e$$
$$= F_{n0} - \frac{B_i^2}{8\pi\mu} + \frac{H_e^2}{4\pi}. \tag{28}$$

From the condition $\partial F_s / \partial\Psi_0 = 0$, we find the equation for Ψ_0 :

$$\left(\frac{B_i}{H_{cl}^{(0)}}\right)^2 = \frac{4\mu\Psi_0^2(\Psi_0^2 - 1)\operatorname{ch}^2(\Psi_0 d/\delta)}{1 - \operatorname{sh}(2\Psi_0 d/\delta)/(2\Psi_0 d/\delta)}. \tag{29}$$

For the critical value of the induction B_{cr} which is determined from the condition $F_s = F_n$, we get

$$\left(\frac{B_{cr}}{H_{cl}^{(0)}}\right)^2 = \frac{\mu\Psi_0^2(2 - \Psi_0^2)}{1 - \operatorname{th}(\Psi_0 d/\delta)/(\Psi_0 d/\delta)}. \tag{30}$$

To find B_{cr} and Ψ_0 (for $B_i = B_{cr}$), we must solve Eqs. (29) and (30).

Equations (29) and (30) differ from those used in Refs. 3 and 4 only by the replacement of H_e and H_{cr} in Refs. 3,4 by $B_i/\sqrt{\mu}$ and $B_{cr}/\sqrt{\mu}$. Moreover, the quantity δ in Eqs. (29) and (30) differs from the δ_0 in Refs. 3,4 by the factor $\mu^{-1/2}$. Therefore it is not necessary to re-investigate Eqs. (29) and (30), and we only recall that for $d < (5/2)\,\delta$ destruction of superconductivity leads to a transition of second order, in which $\Psi_0 = 0$ for the critical field. Here

$$B_{cr}/\sqrt{\mu} = \pm\sqrt{6}\,(\delta/d)\,H_{cl}^{(0)} \equiv \pm H_{cr}^{(0)} \tag{31}$$

where $H_{cr}^{(0)}$ is the critical field for a film with the same values of δ, d and $H_{cl}^{(0)}$ but with $M_0 = 0$ and μ=1.

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*The quantity $H_{cl}^{(0)}$ is the critical field for a large specimen for $M_0 = 0$ and μ=1 (see Ref. 3, where this field is denoted by H_{cl}). We note that the restriction employed in Ref. 3 , which is connected with the consideration only of the region close to T_k , is not essential and it can be discarded.[5]

If $d > (\sqrt{5}/2)\,\delta$, a transition of first order takes place, in which again $B_{cr}/\sqrt{\mu} = \pm H_{cr}^{(0)}$. Inasmuch as $B_{cr} = \mu H_{cr} + 4\pi M_0$, where H_{cr} is the critical value of the external field H_e, we get

$$M_0 > 0: H_{cr}(T) = H_{cr}^{(0)}(T)/\sqrt{\mu} - 4\pi M_0/\mu, \quad (32)$$

$$M_0 < 0: H_{cr}^{\pm}(T) = 4\pi\,|M_0|/\mu \pm H_{cr}^{(0)}(T)/\sqrt{\mu}. \quad (33)$$

Thus precisely the same result is obtained as in the case of large samples, but with the natural replacement of $H_{cl}^{(0)}(T)$ by $H_{cr}^{(0)}(T)$. Therefore there is no need of repeating what was done in Sec. 2. As concerns Figs. 1 and 2, they, being schematic also remain in force, inasmuch as the differences in the temperature dependence of $H_{cr}^{(0)}$ on $H_{cl}^{(0)}$ in this scheme is not essential.* However, in the quantitative relation, the situation changes sharply, for example, for $H_{cl}^{(0)}\,0=400$ Oe $\delta(0) = 5\times10^{-6}$ and $d = 2.5\times10^{-7}$ (the thickness of the film $2d = 50A$), $H_{cr}^{(0)}(0) \approx 20{,}000$ Oe which is already greater than $4\pi M_0$ for large samples of Ni and Co. For very thin films, the value of $4\pi M_0$ must decrease and furthermore, the possibility is not excluded of obtaining superconducting films of 2–3 times greater thinness than those used in the example treated.[6] Further, thin films must be uniform** and possess a relative large coercive force.[7] Therefore use of the configuration with $M_0 < 0$ [see Eq. (33)] can lead to a significant increase in the probability of discovering the existence of superconductivity. Finally, investigation of various ferromagnetic alloys especially with relatively small values of M_0, opens up additional possibilities, in particular if we consider that, for films, by virtue of their lesser density and also because of the small thickness, the value of M_0 should be smaller than for large pieces of metal.

If superconductivity will be discovered in very thin ferromagnetic films, then for further increase in the thickness of the film (for example, as a result of repeated saturation in the metal) the superconductivity decreases, as only $H_{cr}^{(0)}(T)$ remains smaller than $4\pi M_0$ (we are dealing with the case $M_0 > 0$, the value of μ is set equal to unity, as one should expect).

Experiments to show the superconductivity of ferromagnets deserved, we think, a great deal of attention, although it is possible that they will not succeed.

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*Usually it is a satisfactory approximation to assume that

$$H_{Cl}^{(0)}(T) = H_{Cl}^{(0)}(0)\{1-(T/T_{Cr})^2\},$$

$$\delta(T) = \delta(0)\,[1-(T/T_{Cr})^4]^{-1},$$

whence the temperature dependence of $H_{cr}^{(0)}$ in the case (31) is also clear.

**By virtue of the polycrystalline composition of the films, it is perhaps expedient for obtaining sharp pictures to produce saturation of the film in a strong magnetic field.

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SOVIET PHYSICS JETP VOLUME 4, NUMBER 2 MARCH, 1957

Elastic Scattering of Neutrons by 580 MEV Protons

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Differential cross sections were measured for elastic scattering of neutrons by 580 mev protons in the angular region $\vartheta = 35$ to 180° (in the system of the center of inertia) with a 2° angular resolution. The character of the obtained $\sigma_{np}(\vartheta)$ dependency indicates that at ~600 mev there arises in the neutron-proton system a noticeable interaction in states with orbital moments up to $l \approx 6$. The results obtained do not contradict the hypothesis of charge independences.

THERE exists at the present time a sufficiently large number of published papers pertaining to investigation of elastic collisions of neutrons with high energy (100 to 400 mev) protons. However, the known experimental data refer to the energy region where the processes of meson formation are either completely absent, or the total cross section of meson formation in (*n-p*) collisions comprises a very small fraction of the total (*n-p*) interaction cross section. It may be supposed[1,2] that the increased probability of inelastic scattering observed in the region of higher energies* will have a noticeable influence on the character of elastic scattering of neutrons by protons. Experiments in measurements of differential cross sections for elastic scattering of neutrons by 580 mev protons were conducted using the synchrocyclotron of the Institute for Nuclear Problems, Academy of Sciences, USSR, in order to investigate the changes in the process of neutron scattering by protons at nuclear energies which exceed substantially the threshold of pion formation. This communication contains results of these experiments.

1. EXPERIMENTAL ARRANGEMENT

Measurement of differential cross sections in the scattering angle interval $\vartheta = 35.5 - 180°$ (in the center of inertia system) was made by the method of recording the recoil protons from elastic (*n-p*) collisions. Here the difference in the number of recoil protons was determined at angles $\vartheta = 0$ to 70° for paraffin ($CH_{2.09}$) and graphite (C) scatterers placed in the path of the neutron beam. The experimental arrangement is shown in Fig. 1. The neutron beam used in the experiments was obtained as a result of exchange scattering of 680 mev protons with a beryllium target. The energy distribution in the neutron beam had a maximum at 600 mev and a half width of ~130 mev[3]. The average energy of the neutrons for the detector threshold (450 mev) was found to be equal to 580 mev. The intensity of the neutron beam incident on the target was 2×10^4 neutron cm^{-2} sec^{-1}.

Discs of paraffin and graphite of equal stopping power for the recoil protons served as scatterers. The thickness of the scatterers was changed for measurements at different angles and in the recoil angle intervals $\Phi = 0$-15°, 25-60° and 60-70° (referred to the laboratory system) was for graphite scatterer, 5, 3, 0.5 gram/cm^2, respectively.

The detector consisted of three scintillation counters (*1, 2, 3*) connected for coincidence counting. The scintillation counters were operated with photomultipliers type FEU-19 and toluene crystals. The selected geometry insured a 2° angular resolution. The energy threshold of the detector was set by a wolfram or copper filter placed between the two last counters of the detector. The thickness of the filter depended on the scattering angle and was selected to make the detector operate only if the neutron energy exceeded 450 mev. This made it possible to exclude the effect produced by lower energy particles which were present in sufficient quantity in the neutron beam. The counting characteristic of the detector had a plateau of 150-200 volts.

The process of measuring the angular distribution of recoil protons (in relative units) was reduced to determining the number of charged particles which are emitted from the scatterer in a given direction as a result of (*n-p*) collisions, with a subsequent correction for the admixture of mesons and the absorption of protons in the filter of the detector.

The correction for the admixture of charged π-mesons at angles $\Phi \geq 60°$ was determined by means

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* The overall cross section of pion formation at 400 mev is ~ 3% of the total (*n-p*) interaction cross section, and increases to 25-30% at 590 mev[3].

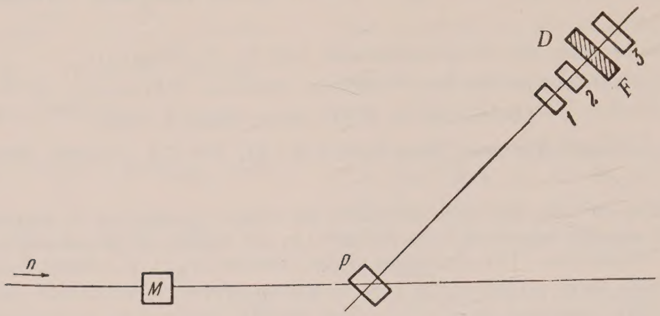

FIG. 1. Arrangement of the apparatus. n — neutron beam; P — tterer; D — detector; *1, 2, 3* — scintillation counters; F — filter; monitor.

of a telescope consisting of three scintillation counters similar to the detector used in measuring the angular distribution of the charged particles. Evaluations made on the basis of data in Ref. 4 showed that if the threshold of the detector is raised above the maximum energy of the recoil protons for a given angle $\Phi \geq 60°$, the detector will register only π — mesons with a registration effectiveness of 70 to 80%. Velocity selections of particles was used for separation of π — mesons in measurements at angles $\Phi < 60°$. For this, the first counter of the detector was replaced by a Cerenkov counter. Plexiglass was used as the radiator for the Cerenkov counter at angles $\Phi = 30$ to $60°$ and water for angles $\Phi < 30°$. The correction was determined by comparing the number of particles registered by the detector under ordinary conditions in corresponding experiments on (n-p) scattering cross section measurements with the number of particles registered by the detector when the first counter is replaced by a Cerenkov counter. The effectiveness of the Cerenkov counter was taken as 80%[5]. It should be noted that at proton recoil angles $\Phi < 45°$ the maximum energy recoil protons could also have been registered by the Cerenkov counter. For this reason, when making measurements at the indicated angles, a portion of the filter used to lower the energy of the protons to the threshold value of the Cerenkov counter was placed in front of the detector.

To determine the corrections for nuclear absorption by the method of coordinated telescopes* there were registered the events of (p-p) collisions taking place in the scatterer placed in the 667 mev proton beam. The correction was determined as the ratio of (p-p) collisions recorded when the detecting telescope is operating without a filter to the number of registered collisions when a filter of a given thickness is present in the telescope. The geometry of the detector used in these experiments corresponded exactly to the experimental conditions when making measurements of differential (n-p) scattering cross sections.

The angle at which the telescope was set was selected so that the energy of scattered protons entering the telescope was equal to the average energy of the recoil proton at the angle for which the absorption correction is being measured.

Absolute values of (n-p) scattering differential cross sections were determined by normalizing the obtained angular distribution of the recoil protons to the complete cross section of elastic scattering of neutrons by protons. The value of this cross section for 580 mev was obtained from the data contained in Ref. 6 as the difference between the total cross section for (n-p) interaction $\sigma_t(np) = (36 \pm 2) \times 10^{-27}$ cm^2 and the overall cross section for meson formation in (n-p) collisions $\sigma_\pi(np) = (10 \pm 2) \times 10^{-27}$ cm^2, which amounts to $(26.0 \pm 3) \times 10^{-27}$ cm^2. For normalization it was assumed that in the angular region $\vartheta < 35°$ (center of inertia system) the differential (n-p) scattering cross section is substantially constant in magnitude and is equal to the value of the cross section at 35° just as in the case of lower energies (300-400 mev) where it has been confirmed by experiments[7].

It should be noted, however, that deviation of the true distribution from that assumed in normalizing cannot significantly distort the obtained results, since scattering in the angular region $0 \leq \vartheta < 35°$

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* A method in which two telescopes are connected in coincidence and register simultaneously the scattered particle and the recoil particle.

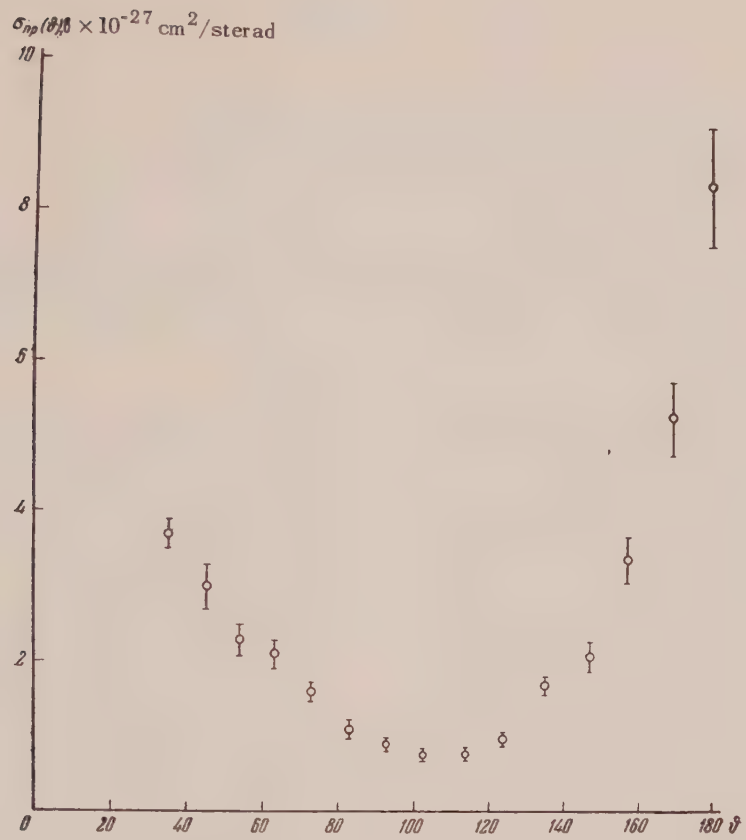

FIG. 2. Dependence of the differential cross section of angular (n-p) scattering on the scattering angle (in the center of inertia system).

constitutes only a very small portion of the total of (n-p) elastic scattering cross section. If we assume the $\sigma_{np}(\vartheta)$ curve to be symmetrical about the 90° point, this fraction does not exceed 20%.

2. RESULTS OF MEASUREMENTS

Results of the measurements are shown in Fig. 2. The errors shown on the graph are the standard statistical errors in the determination of angular distribution for recoil protons. Not included are the errors in determining the total (n-p) elastic scattering cross section (12%) and some uncertainty in the normalization (about 10%) due to the absence of accurate data on the $\sigma_{np}(\vartheta)$ dependence in the region $\vartheta < 35°$.

The dependence of the obtained differential cross section on the scattering angle indicates that at 580 mev as well as in the region of lower energies, exchange forces play an important role in the (n-p) interaction. The contributions of the usual and exchange interactions to the total elastic scattering cross section are of the same order. The results also show that, with increase of energy, the nature of the scattering changes. The anisotropy of the scattering is increasing, the ratio of the cross sections $\sigma_{np}(180°)$ to $\sigma_{np}(90°)$ reaches the value 9 ± 1.2. The asymmetry of the curve $\sigma_{np}(\vartheta)$ about the angle $\vartheta = 90°$ is clearly noticeable even in the angular region very close to $\vartheta = 90°$.

Analysis of obtained results permits us to make the following conclusions:

1. The results do not contradict the hypothesis of isotopic invariance. The values of differential scattering cross sections $\sigma_{np}(90°)$ and $\sigma_{pp}(90°)$ [8] for 580 mev nucleons satisfy sufficiently close the known relation $4\sigma_{np}(90°) \geq \sigma_{pp}(90°)$ which follows from the above hypothesis;

2. The contributions of the states of the (n-p) system with isotopic spins $T = 0$ and $T = 1$ to the scattering cross section at $\vartheta = 90°$ are of the same order of magnitude [$\sigma_{T=1}(90°) = 3 \times 10^{-27}$; $\sigma_{T=0}(90°) = 1 \times 10^{-27}$ cm^2 sterad^{-1}]. This is possibly

connected with the existence in both states ($T = 0$ and $T = 1$) of a very strong interaction. It should be noted, however, that convincing proof in favor of this assumption can, seemingly, be obtained only by performing a complete phase analysis;

3. The noticeable asymmetry of the curve $\sigma_{np}(\vartheta)$ about the angle 90° indicates that interference between the waves corresponding to states $T = 0$ and $T = 1$ exerts a considerable influence on the nature of the scattering. It is interesting to note that at the neutron energy of 90 mev[9] where, according to the angular distribution of $\sigma_{np}(\vartheta)$, it can be concluded that scattering is described by S, P and D waves, interference between waves corresponding to states $T = 0$ and $T = 1$ is practically absent, and the curve $\sigma_{np}(\vartheta)$ is symmetrical about the angle $\vartheta = 90°$[10]. Thus, the asymmetry of the curve is apparently the result of interaction between two nucleons in system states characterized by high orbital moments $l > 2$;

4. The absence of a relativistic theory of scattering does not permit a strict interpretation of the obtained data. In the nonrelativistic approximation, however, it can be shown[11] that the highest number of the Legendre polynomial entering into the angular distribution is equal to $2l_{\max}$, where $l_{\max}$ denotes the maximum value of the orbital moment of motion corresponding to the highest state in which, for a given nucleon energy, noticeable interaction takes place. Computations show that the angular distribution $\sigma_{np}(\vartheta)$ can be approximated by the expression $\sigma_{np}(\vartheta) = \bar{\lambda}^2[(1.29 \pm 0.05) + (0.24 \pm 0.13)P_1 + (1.42 \pm 0.2)P_2 - (0.87 \pm 0.24)P_3 + (0.28 \pm 0.22)P_4 - (0.5 \pm 0.16)P_5 + (0.14 \pm 0.1)P_6 - (0.33 \pm 0.1)P_7 + (0.19 \pm 0.18)P_8 - (0.2 \pm 0.24)P_9 + (0.39 \pm 0.26)P_{10} + (0.1 \pm 0.19)P_{11} + (0.25 \pm 0.15)P_{12}]$, where $P_{1,2\ldots}$ are Legendre polynomials. Thus, the maximum value of the orbital momentum corresponds to $l_{\max} \approx 6$. If we now make use of the target parameter concept, it is possible to evaluate roughly the effective size of the interaction region. An evaluation yields for the value of r 2.2×10^{-13} cm.

In conclusion, the authors wish to express their thanks to Prof. V. P. Dzhelepov for his supervision of this work and to Prof. M. G. Meshcheriakov for his discussion of and critical comments on the results. The authors are also very thankful to Miss G. H. Tentiukova for making the required computations.

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Determination of the Branching Ratio in the Disintegration Scheme of Po^{210}

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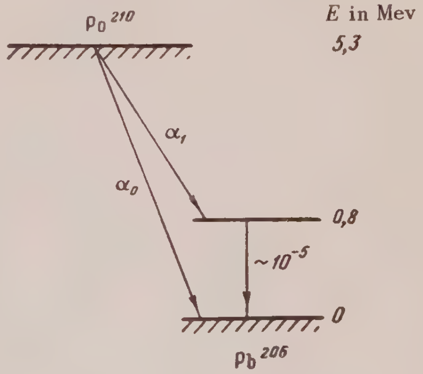
Measurements were made of the branching ratio in the disintegration scheme of Po^{210}. The probability of transition to the excited level Pb^{206} (with an energy of $E = 800$ kev) was found to be $1.2 \pm 0.2 \times 10^{-5}$.

1. FOR a long time after the discovery of Po^{210} it was considered that this radioactive isotope was a monochromatic α – radiator which did not emit any other radiation besides 5.3 mev α– particles. Only in 1930 Bothe and Becker[1], while investigating γ-radiation arising under the action of α-rays from Po^{210}, discovered that the same isotope emits a characteristic, very weak, γ-radiation. A comparison with a Ra standard enabled the authors to obtain a rough evaluation of the absolute amount of this radiation ($\sim 10^{-5}$ quanta/α-disintegration Po^{210}). During succeeding years the radiation of Po^{210} has not been the subject of study by experimental physicists. With the exception of the works of Curie and Joliot[2] and Bothe[3], whose results were not too clear nor reliable, nothing has been published in literature concerning γ–rays of Po^{210} for 15 years following the discovery of Bothe and Becker. The question of γ–radiation from Po^{210} was again raised in 1946 in connection with the publication of the work of Chang[4] who reported that there seems to exist a "thin structure" spectrum of Po^{210}. Chang found in the energy region 3.5 to 5.3 mev 12 lines of intensities from 1.3×10^{-4} to 2×10^{-5} from the basic 5.3 mev line. These results of Chang naturally aroused special interest concerning γ–radiation of Po^{210}, since they contradicted the little information on the intensity and energies of γ–rays from Po^{210} available in physics at that time. There appeared, therefore, after Chang's communication, a series of papers[5-7] on the investigation of energy and intensity of γ–radiation from Po^{210}. It was concluded in these investigations that the energy of the nuclear radiation is 700 to 800 kev and its emission does not exceed 2×10^{-5} γ–quanta/α–disintegration of Po^{210}. That the results of Chang were faulty was proven by the work of Wadey[8] and subsequent experiments of Haissinsky[9]. The appearance of short path α–particles was explained by the diffusion of the Po^{210} atoms in the metal backing.

In recent years there appeared several more papers[10-15] on the study of γ–radiation from Po^{210}, its energy, absolute intensity, conversion coefficients and their relations for different shells. Results obtained by different authors are shown in Table I.

The work of Benedetti and Minton deserves special attention. The authors were first to observe, by means of scintillation counters, short range α–particles coincident with the ~ 803 kev γ–rays. The difference between the energy of the basic group α–particles of Po^{210} and that of the short range line corresponded, within the limits of experimental error, with the energy of the γ–rays. The shape of the angular correlation curve between the short range alpha-particles and the γ–rays $(I_\gamma \sim A \sin^2 2\alpha)$ and the lifetime of the excited state of Pb^{206} ($< 10^{-9}$ sec) indicated the presence of an electric quadrupole γ–transition. Thus we can accept the disintegration scheme for Po^{210} shown in the diagram.



The intensity of γ–radiation from Po^{210} and, consequently the branching ratio in the disintegration scheme of this isotope, were not reliably determined at that time. Results obtained by different authors vary considerably. Thus the following values were obtained: in Ref. 3, 0.8 to 1×10^{-5};

TABLE I

γ– Radiation Po^{20}

Author	Year of the work	Energy E_γ (kev)	Emission n_γ $\frac{\text{quanta}}{\text{disint.}}$	Conversion coefficient α_K (%)	α_K/α_L
Alburger and Friedlander[10]	1951	800±6		<5	3.7±0.5
Grace et al.[11]	1951		1.80±0.14		
Barber and Helm[12]	1952		1,5±0.4	~12	
Pringle, Taylor et al.[13]	1952	804		~20—30	
Benedette and Minton[14]	1952	803 ($E_{\alpha_1} = 4500$)			
Rioux[15]	1952		1.6±0.2	~9	
Present work	1954		1,2±0,2		
Hayward, Hoppe and Mann[16]	1955		1.22±0,06		

in Ref. 11, $1.8 \pm 0.14 \times 10^{-5}$; in Ref. 12, $1.5 \pm 0.4 \times 10^{-5}$; in Ref. 15, $-1.6 \pm 0.2 \times 10^{-5}$. From the data of Benedetti and Minton[14] one can obtain the value 0.5×10^{-6}. A more or less accurate measurement of the number of γ – quanta in the disintegration of Po^{210} would be of interest from the standpoint of clarifying the multipole nature of this transition, since at the present time no definite conclusion can be made on this question.

2. To determine the value of the branching ratio in the disintegration scheme of Po^{210} it is obviously necessary to determine, in addition to the disintegration rate of any single source of γ – quanta, also the absolute activity (number of disintegrations per second). In the case of polonium preparations of high activity (> 10 to 100 millicuries) the usually employed methods of α–measurements---ionization or impulse chamber and the luminescent counter method---can result in considerable errors not only due to diffusion and migration of active atoms but also due to the absorption of particles in the active layer itself and their reflection from the backing material, appearance of counts of not readily understood origin, etc. Besides, as we have shown, in the case of strong open sources of Po^{210}, chemical-physical processes take place which are apparently connected with the formation of surface films of noticeable thickness. This phenomenon can distort to a considerable degree the values of activity for polonium sources obtained by the above-mentioned methods. A comparison, carried out by us, of results of measuring activities of a series of strong Po^{210} sources in the calorimeter, in a luminescent arrangement and in an impulse chamber of a small solid angle, showed that α – measurements by the last two methods can give considerable errors reaching sometimes 30 to 50%. It is possible that the inaccuracy in the determination of the polonium content in the source is the cause of the diversity in the data on emission of γ – quanta from Po^{210} mentioned above.

3. The source used by us was a preparation of pure Po^{210}. Absence of any noticeable radioactive impurities was confirmed by long time (eight months duration) calorimetric measurements of its disintegration curve as well as of the absorption curve of its γ – radiation in lead. The decrease of the activity of the source also corresponded to the half life of Po^{210} (138.5 days).

The number of γ – quanta emitted by the source in unit time was determined by comparison with a source of Co^{60} of known activity ($N_0 = 19.3 \pm 0.8 \times 10^3$ disint/sec). These measurements were made with a standard β – counter B-1 with an additional 2 mm aluminum filter. Sources of Po^{210} and Co^{60} were placed at an accurately determined distance from the counter, and the two sources were, in addition, surrounded by 1 mm layer of lead. This was done to remove soft radiation of Po^{210} if present. This radiation, as measurements have shown, is not stopped by the 2mm alluminum filter. The results of two series of similar measurements are shown in Table II. Determination of the absolute activity of the source was made in a double static calorimeter used in the Radium Institute, Academy of Sciences, USSR, for the measurement of absolute activities of radioactive preparations[17].

Results of these measurements are also shown in Table II. In processing the calorimetric data, the quantity of heat emitted by one Curie of Po^{210} in the calorimeter was taken as 32.01×10^{-3} watts, or 27.54 m cal/hr[$E_{\alpha_0}(Po^{210}) = 5.298$ mev; one Curie = 3.7×10^{10} disint/sec]. Knowing the ratio

TABLE II

Measurement of γ-Radiation Intensity and Activity of the Po^{210} Source

Date	γ-measurements			Calorimetric measurements	
	Number of pulses per minute N_γ		$k_0 = \frac{N_\gamma P(o^{210})}{N_\gamma (Co^{60})}$	Quantity of heat Q, cal/hour	Activity of source N_α, mCu
	Po^{210}	Co^{60} standard			
11.VII				4.19	152.0
12.VII	407±6	402±6	1.013±0.035		
14.VII	427±6	418±6	1.021±0.035	4.15	150.8
2.VIII				3.73	135.3
9.VIII				3.58	130.1
1.X				2.77	100.5
2.X				2.77	100.5

$\bar{k}_0 = 1{,}017 \pm 0.035$ Value of N_α averaged 150.3 mCu for 13. VII

k_0 of the γ-ray intensity of the calibrating source to that of the measured Po^{210} preparation and their absolute activities N_0 and N_α, it was possible to find n_γ---the number of 800 kev γ-quanta emitted from Po^{210} per each α-disintegration. It was thereby necessary to introduce also correction coefficients for the difference in the sensitivity of our counter to γ-radiation from Co^{60} and Po^{210} —k_1; for the difference in absorption of these radiations in the millimeter lead foil surrounding both counters—k_2; and in the 2 mm. aluminum filter—k_3. The experimentally obtained values of these coefficients were equal respectively to: 1.56 ± 0.08; 1.043 ± 0.005 and 1.014 ± 0.005, while the value obtained for n_γ was

$$N_\gamma = (2N_0/N_\alpha)\cdot k_0 \cdot k_1 \cdot k_2 \cdot k_3 = 1.16 \times 10^{-5}$$

$$\gamma\text{-quanta}/\alpha\text{-disint Po}^{210}$$

with a maximum possible error ±14%. Considering the internal conversion of Po^{210} γ-radiation (E_γ = 800 kev) in the K- and L-shells (1.07% [10,18]) and rounding off the value of N_γ and its error we have finally:

$$N = 1.2 \pm 0.2 \times 10^{-5}\ \gamma\text{-transitions}/\alpha\ \text{disint Po}^{210}.$$

The values of the absolute number of γ-transitions in Po^{210}, and therefore the values of the branching ratio in the disintegration scheme of this isotope obtained by us were considerably less than those reported in Refs. 11, 12 and 15. Since the usual ionization or impulse chamber methods were used by these investigators to determine the activity of the polonium source, it was possible that some of the α-particles were not registered, as mentioned above, due to the diffusion of the active atoms into the backing material and absorption in the surface films. This could result in raising the value obtained for the absolute intensity of nuclear radiation from Po^{210}. It should be mentioned that after the completion of our measurements there appeared in print a paper by Hayward, Hoppes and Mann[16] (the results contained in this paper were entered into Table I). The authors, who used a balanced calorimeter to measure the absolute activity of the polonium preparation, obtained a value for the γ-ray emission very close to ours, namely, 1.22×10^{-5}. Measurement of the γ-ray intensity was made by a comparison with a Co^{60} standard, using a scintillation counter with a NaI(Tl) crystal.

In conclusion it should be noted that the values obtained by different authors for the branching ratio in the Po^{210} disintegration scheme do not agree with the deductions of Benedetti and Minton concerning the quadrupole nature of γ-radiation from Po^{210}. Indeed, the penetration ratio for α-particles 5.3 and 4.5 mev and $\Delta j = 2$ computed according to Gamow's theory as 3.2×10^{-5}, is considerably larger than the experimental values for this ratio. The experimental value of the ratio of conversion coeffecients in the K and L shells, $\alpha_K/\alpha_L = 3.7 \pm 0.5$, obtained by Alburger and Friedlander[10] is evidence against the quadrupole nature of the γ-transition in the disintegration scheme of Po^{210}.

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Theory of Classical Systems of Interacting Particles Obeying a Noncentral Interaction Law. I.

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The theory of systems of interacting particles with noncentral interaction law is considered on the basis of Bogoliubov's method.[1] Successive approximations are obtained for the distribution functions in the two simplest cases: for a gas consisting of axially symmetric diatomic neutral molecules and for a dipole crystal [provided the condition (32) holds].

THE theory of systems with a noncentral interaction law between the particles is characterized by a number of special difficulties; however, in certain cases it can be constructed rather easily on the basis of Bogoliubov's method.

Let us consider the simplest possible case of a classical system, in which the interaction law between the particles differs only slightly from the law of central short range forces. Such is the case for systems consisting of neutral diatomic molecules when their weak orientation interaction with each other is taken into account.[2]

We introduce the distribution function F_s so normalized that

$$(4\pi V)^{-s} F_s(\mathbf{q}_1, \ldots, \mathbf{q}_s; \vartheta_1, \varphi_1, \ldots, \vartheta_s\varphi_s) \qquad (1)$$
$$\times d\mathbf{q}_1 \ldots d\mathbf{q}_s d\Omega_1 \ldots d\Omega_s$$

determines the probability of the location of the centers of mass of a chosen group of s molecules in the respective volumes $d\mathbf{q}_1, \ldots, d\mathbf{q}_s$ for orientations of these molecules determined by the elements of solid angle $d\Omega_1, \ldots, d\Omega_s$, where $\mathbf{q}_1, \ldots, \mathbf{q}_s$ are vectors determining the position of the centers of mass of the molecules and the angles $\vartheta_1, \varphi_1 \ldots \vartheta_s, \varphi_s$ determine the orientation of the axes of the molecules relative to the chosen system of coordinates. The distribution functions thus determined characterize a system consisting of identical, symmetric diatomic molecules and satisfy the equality

$$F_s(\mathbf{q}_1 \ldots \mathbf{q}_s; \vartheta_1, \varphi_1 \ldots \vartheta_s, \varphi_s)$$
$$= (4\pi V)^s \int D_N d\mathbf{q}_{s+1} \ldots d\mathbf{q}_N d\Omega_{s+1} \ldots d\Omega_N;$$
$$D_N(\mathbf{q}_1, \ldots \mathbf{q}_N; \vartheta_1, \varphi_1, \ldots \vartheta_N, \varphi_N)$$
$$= Q_N^{-1} \exp(-U_N/\Theta)$$

is a Gibb's configuration function and

$$U_N = \sum_{1 \leqslant i < j \leqslant N} L_{ij}(\mathbf{q}_i, \mathbf{q}_j; \vartheta_i, \varphi_i; \vartheta_j, \varphi_j),$$

where L_{ij} is the interaction potential and

$$Q_N = \int \exp(-U_N/\Theta)\, d\mathbf{q}_1 \ldots d\Omega_N; \quad \Theta = kT.$$

Beginning with the identities

$$\partial D_N/\partial q_1^\alpha + (D_N/\Theta)\, \partial U_N/\partial q_1^\alpha = 0, \quad (\alpha = 1, 2, 3),$$
$$\partial D_N/\partial \vartheta_1^\alpha + (D_N/\Theta)\, \partial U_N/\partial \vartheta_1^\alpha = 0, \quad (\alpha = 1, 2),$$

in the usual way, considering equalities of the type

$$\int \frac{\partial L_{1,s+1}}{\partial q_1^\alpha} F_{s+1}$$
$$(\mathbf{q}_1 \ldots \mathbf{q}_s, \mathbf{q}_{s+1}; \vartheta_1, \varphi_1 \ldots \vartheta_s, \varphi_s, \vartheta_{s+1}, \varphi_{s+1})$$
$$\times d\mathbf{q}_{s+1} d\Omega_{s+1}$$
$$= \int \frac{\partial L_{1,s+2}}{\partial q_1^\alpha} F_{s+1}$$
$$(\mathbf{q}_1 \ldots \mathbf{q}_s, \mathbf{q}_{s+2}; \vartheta_1, \varphi_1 \ldots \vartheta_s, \varphi_s, \vartheta_{s+2}, \varphi_{s+2})$$
$$\times d\mathbf{q}_{s+2} d\Omega_{s+2},$$

we get to the equations

$$\frac{\partial F_s}{\partial q_1^\alpha} + \frac{1}{\Theta}\frac{\partial U_s}{\partial q_1^\alpha} F_s + \frac{1}{4\pi\Theta v}\iint \frac{\partial L_{1,s+1}}{\partial q_1^\alpha} F_{s+1} d\mathbf{q}_{s+1} d\Omega_{s+1}$$
$$= 0, \quad \alpha = 1, 2, 3.$$

$$\frac{\partial F_s}{\partial \vartheta_1^\alpha} + \frac{1}{\Theta}\frac{\partial U_s}{\partial \vartheta_1^\alpha} F_s \qquad (2)$$
$$+ \frac{1}{4\pi\Theta v}\iint \frac{\partial L_{1,s+1}}{\partial \vartheta_1^\alpha} F_{s+1} d\mathbf{q}_{s+1} d\Omega_{s+1} = 0, \quad \alpha = 1, 2$$

($v = V/N$). We limit ourselves here to a consideration of an interaction potential of the following type:

$$L_{1,2} = \Phi(|\mathbf{q}_1 - \mathbf{q}_2|) \qquad (3)$$
$$\times [1 + \lambda\psi_{1,2}(\vartheta_1, \varphi_1, \vartheta_2, \varphi_2)],$$

where Φ is the interaction potential which corresponds to short range central forces and $\psi_{1,2}$ are terms taking into account the dependence of the interaction on the orientation of the particles. Although we have written explicitly only the dependence of $\psi_{1,2}$ on the pairs of variables ϑ_1, φ_1 and ϑ_2, φ_2 which determine the orientation of the axes of the molecules in space, this function actually depends on the pairs of variables $\vartheta_{12}, \varphi_{12}$ also. These latter determine the orientation of the line of centers of the pair of particles under consideration; λ denotes the parameter whose smallness determines the weakness of the orienting effects in the system under consideration.[2]

Limiting ourselves to a consideration of systems with sufficiently low density, we employ an expansion in powers of the latter:

$$F_s = F_s^0 + v^{-1} F_s^1 + v^{-2} F_s^2 + \dots \tag{4}$$

The corresponding groups of the systems of equations will have the form

$$\frac{\partial F_s^0}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s}{\partial q_1^\alpha} F_s^0 = 0 \qquad (\alpha = 1,\ 2,\ 3), \tag{5}$$

$$\frac{\partial F_s^0}{\partial \vartheta_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s}{\partial \vartheta_1^\alpha} F_s^0 = 0 \qquad (\alpha = 1,\ 2);$$

$$\frac{\partial F_s^1}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s}{\partial q_1^\alpha} F_s^1 + \frac{1}{4\pi\Theta} \tag{6}$$

$$\times \iint \frac{\partial}{\partial q_1^\alpha} \Phi_{1,s+1} (1 + \lambda \psi_{1,s+1}) F_{s+1}^0 d\mathbf{q}_{s+1} d\Omega_{s+1} = 0,$$

$$\frac{\partial F_s^1}{\partial \vartheta_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s}{\partial \vartheta_1^\alpha} F_s' + \frac{1}{4\pi\Theta}$$

$$\times \iint \frac{\partial}{\partial \vartheta_1^\alpha} \Phi_{1,s+1} (1 + \lambda \psi_{1,s+1}) F_{s+1}^0 d\mathbf{q}_{s+1} d\Omega_{s+1} = 0$$

and so forth. Here,

$$U_s = \sum_{1 \leqslant i < j \leqslant s} \Phi_{ij} (|\mathbf{q}_i - \mathbf{q}_j|) \tag{7}$$

$$\times (1 + \lambda \psi_{ij} (\vartheta_i, \varphi_i, \vartheta_j, \varphi_j))$$

can be written in the form of a sum $U_s = U_s^0 + \lambda U_s^1$, where

$$U_s^0 = \sum_{1 \leqslant i < j \leqslant s} \Phi (|\mathbf{q}_i - \mathbf{q}_j|),$$

$$U_s^1 = \sum_{1 \leqslant i < j \leqslant s} \Phi (|\mathbf{q}_i - \mathbf{q}_j|) \psi_{ij} (\vartheta_i, \varphi_i, \vartheta_j, \varphi_j).$$

Now, for each of the systems, we consider an expansion in powers of the small parameter λ :

$$F_s^i = F_s^{i0} + \lambda F_s^{i1} + \lambda^2 F_s^{i2} + \dots, \tag{8}$$

$$i = 0,\ 1, \dots$$

The corresponding systems of equations in this case are described in the following way:

$$\frac{\partial F_s^{00}}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^0}{\partial q_1^\alpha} F_s^{00} = 0, \qquad \frac{\partial F_s^{00}}{\partial \vartheta_1^\alpha} = 0 \tag{5a}$$

$$\frac{\partial F_s^{01}}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^0}{\partial q_1^\alpha} F_s^{01} + \frac{1}{\Theta} \frac{\partial U_s^1}{\partial q_1^\alpha} F_s^{00} = 0, \tag{5b}$$

$$\frac{\partial F_s^{01}}{\partial \vartheta_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^1}{\partial \vartheta_1^\alpha} F_s^{00} = 0 \quad \text{etc.} \tag{6a}$$

$$\frac{\partial F_s^{10}}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^0}{\partial q_1^\alpha} F_s^{10} \tag{6b}$$

$$+ \frac{1}{4\pi\Theta} \iint \frac{\partial \Phi_{1,s+1}}{\partial q_1^\alpha} F_{s+1}^{00} d\mathbf{q}_{s+1} d\Omega_{s+1} = 0; \quad \frac{\partial F_s^{10}}{\partial \vartheta_1^\alpha} = 0;$$

$$\frac{\partial F_s^{11}}{\partial q_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^0}{\partial q_1^\alpha} F_s^{11} + \frac{1}{\Theta} \frac{\partial U_s^1}{\partial q_1^\alpha} F_s^{10}$$

$$+ \frac{1}{4\pi\Theta} \iint \frac{\partial \Phi_{1,s+1}}{\partial q_1^\alpha} F_{s+1} d\mathbf{q}_{s+1} d\Omega_{s+1}$$

$$+ \frac{1}{4\pi\Theta} \iint \frac{\partial}{\partial q_1^\alpha} (\Phi_{1,s+1} \psi_{1,s+1}) F_{s+1}^{00} d\mathbf{q}_{s+1} d\Omega_{s+1} = 0,$$

$$\frac{\partial F_s^{11}}{\partial \vartheta_1^\alpha} + \frac{1}{\Theta} \frac{\partial U_s^1}{\partial \vartheta_1^\alpha} F_s^{10}$$

$$+ \frac{1}{4\pi\Theta} \iint \frac{\partial}{\partial \vartheta_1^\alpha} (\Phi_{1,s+1} \psi_{1,s+1}) F_{s+1}^{00} d\mathbf{q}_{s+1} d\Omega_{s+1} = 0$$

and so on. The solution of the system (5a) is, evidently,

$$F_s^{00} = C_s \exp(-U_s^0/\Theta), \tag{9}$$

where C_s is a constant which is determined by the condition of the decay of the correlation

$$F_s(\mathbf{q}_1 \dots \mathbf{q}_s;\ \vartheta_1, \varphi_1, \dots \vartheta_s, \varphi_s) \tag{10}$$

$$- \prod_{1 \leqslant i \leqslant s} F_1(\hat{\mathbf{q}}_i, \vartheta_i, \varphi_i) \to 0$$

for all $|\mathbf{q}_i - \mathbf{q}_j| \to \infty$ and the normalization condition

$$\lim_{V\to\infty} \frac{1}{4\pi V} \iint F_1(\mathbf{q},\vartheta,\varphi)\, d\mathbf{q}\, d\Omega = 1. \qquad (11)$$

We find from the second of the equations of (5b) that

$$F_s^{01} = -(U_s^1/\Theta)\, F_s^{00}. \qquad (12)$$

This expression automatically satisfies the first equation of (5b).

The general solution of (5b) can be written formally as

$$F_s^{01} = -(U_s^1/\Theta) F_s^{00} + C g_s(\mathbf{q}_1 \ldots \mathbf{q}_s),$$

where g_s is determined upon substitution in the first equation of (5b) and is equal to $g_s = F_s^{00}$. However, it is easy to prove directly that preservation in the solution of terms of the type $Cg_s(\mathbf{q}_1, \ldots \mathbf{q}_s)$, both in (12) and in all subsequent approximations, would reduce to the corresponding determination of the normalization constant; in what follows, we shall neglect these terms.

Proceeding stepwise from one system of equations to the other, we find

$$F_s^{02} = {}^1\!/_2 (U_s^1/\Theta)^2 F_s^{00}. \qquad (13)$$

Further,

$$F_s^{10} = \exp(-U_s^0/\Theta) \qquad (14)$$

$$\times \int \Big\{ \prod_{1\leqslant i\leqslant s} (1 + f(|\mathbf{q}_i - \mathbf{q}_{s+1}|)) - 1 - \sum_{1\leqslant i\leqslant s} f(|\mathbf{q}_i - \mathbf{q}_{s+1}|) \Big\}\, d\mathbf{q}_{s+1},$$

$$F_s^{11} = -\frac{U_s^1}{\Theta} F_s^{10} \qquad (15)$$

$$- \frac{1}{4\pi\Theta} \sum_i \iint \Phi_{i,s+1} \psi_{i,s+1} F_{s+1}^{00}\, dq_{s+1}\, d\Omega_{s+1}$$

$$+ a_s^{11} \exp(-U_s^0/\Theta),$$

$$F_s^{12} = -\frac{U_s^1}{\Theta} F_s^{11} - \frac{1}{2}\left(\frac{U_s^1}{\Theta}\right)^2 F_s^{10} \qquad (16)$$

$$+ \frac{1}{4\pi\Theta^2} \iint \frac{1}{2}\Big(\sum_i \Phi_{i,s+1}\psi_{i,s+1}\Big)^2 F_{s+1}^{00}\, d\mathbf{q}_{s+1}\, d\Omega_{s+1}$$

$$+ a_s^{12} \exp(-U_s^0/\Theta),$$

where

$$f(|\mathbf{q}_i - \mathbf{q}_{s+1}|) = \exp(-\Phi(|\mathbf{q}_i - \mathbf{q}_{s+1}|)/\Theta) - 1;$$

a_s^{11} and a_s^{12} are constants. Carrying forward the computation, it is easy to determine a much higher approximation. Knowledge of the first terms of the series described here is sufficient that we can, using unitary and binary distribution functions, estimate the effect of noncentral forces on the equilibrium properties of the system under consideration.

Calculation of the mean and free energies of the system can be carried out in the usual fashion and can be compared with the results of the corresponding calculation for purely central interaction on the one hand, and with experimental data for diatomic gases on the other.

As a second problem of known simplicity, we consider a molecular crystal consisting of molecules which possess a permanent dipole moment.*

Let us consider a dipole crystal of cubic structure, at the lattice points of which are located dipole molecules with moments $\mathbf{p}_i$ $(i = 1, \ldots, N)$.

The set of angular coordinates which determine the orientation of each dipole (ϑ_i, φ_i) will be denoted by θ_i, the interaction potential of two dipoles $\Phi_{a_i a_k}$ is written in the form

$$\Phi_{a_i a_k} = \mathbf{p}_{a_i} \mathbf{p}_{a_k} r_{a_i a_k}^{-3} - 3(\mathbf{p}_{a_i} \mathbf{r}_{a_i a_k})(\mathbf{p}_{a_k} \mathbf{r}_{a_i a_k})\, r_{a_i a_k}^{-5}, \qquad (17)$$

where the indices a_i denote the types of molecules. The type characteristics are introduced in the following way. The dipole crystal, in which the centers of mass of the dipole molecules are rigidly fixed at the lattice points, will be considered as a system of particles of different types. The type of particle is determined by the lattice vector of that point at which the center of mass of the molecule is located. In Eq. (17) the type characteristics appear in the form of numerical factors in connection with the equality

$$r_{a_i a_k} = |\mathbf{r}_{a_i} - \mathbf{r}_{a_k}| = d l_{a_i a_k}^{1/2}, \qquad (18)$$

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*The case of a gas consisting of molecules with constant quadrupole moments can easily be considered if the law of interaction of two quadrupoles is taken in the form

$$\Phi_{ab} = (3Q_a Q_b/16 r^5)$$

$$\times [1 - 5\cos^2\vartheta_a - 5\cos^2\vartheta_b - 15\cos^2\vartheta_a \cos^2\vartheta_b + 2(\sin\vartheta_a \sin\vartheta_b \cos(\varphi_a - \varphi_b) - 4\cos\vartheta_a \cos\vartheta_b)^2],$$

where Q is the quadrupole moment of a cylindrically symmetrical molecule.

where d is the lattice constant, l is an integer. They also follow from the cosines of the angles which determine the direction of the lines of centers of the two dipoles under consideration. In such a model the dipole crystal is an "orientation" gas or liquid at the temperature which exceeds the temperature of orientation flow.

We introduce the distribution function $F_{a_1 \dots a_s}$ so normalized that

$$(4\pi)^{-s} F_{a_1 \dots a_s}(\theta_1, \dots \theta_s)\, d\Omega_1 \dots d\Omega_s,$$

$$d\Omega_i = \sin \vartheta_i d\vartheta_i d\varphi_i,$$

is the probability that the direction of the dipole moments in the chosen group of s-dipoles lies within elements of solid angles $d\Omega_1 \dots d\Omega_s$ for arbitrary orientations of the other dipoles.

Considering the functional defined on a class of arbitrary regular functions $U_1(\theta_1) \dots U_N(\theta_N)$ given on the surface of the unit sphere

$$L_N(U_1, \dots U_N) \tag{20}$$

$$= \int D_N \prod_{1 \leqslant a_i \leqslant N} (1 + 4\pi U_{a_i}(\theta_i))\, d\Omega_1 \dots d\Omega_N,$$

and making use of the identity

$$\frac{\partial D_N}{\partial \theta_i^\alpha} + \frac{\partial U_N}{\partial \theta_i^\alpha} \frac{D_N}{\mu} = 0, \quad U_N = \frac{1}{2} \sum_{a_i, a_j}^{N} \Phi_{a_i a_j},$$

(where $\mu = kT$, $\alpha = 1, 2$, $\theta_i^1 = \vartheta_i$ and $\theta_i^2 = \varphi_1$) it is easy to express the equations in functional derivatives

$$\frac{\partial}{\partial \theta_i^\alpha} \frac{\delta L}{\delta U_{a_1 \theta_1}} + \frac{1}{4\pi\mu} \tag{21}$$

$$\int \sum_{1 \leqslant a_2 \leqslant N} \frac{\partial \Phi_{a_1 a_2}}{\partial \theta_1^\alpha} \frac{\delta^2 L}{\delta U_{a_1 \theta_1} \delta U_{a_2 \theta_2}} (1 + 4\pi U_{a_2})\, d\Omega_2 = 0.$$

Carrying out the functional differentiation and replacing in the final calculation the derivative functions U_a by zero, we obtain a system of equations for the distribution functions introduced above [taking into account the relations

$$(\delta^s L / \delta U_{a_1 \theta_1} \dots \delta U_{a_s \theta_s})_0 = F_{a_1 \dots a_s}(\theta_1 \dots \theta_s].\tag{22}$$

this system can be written in the form

$$\frac{\partial F_{a_1 \dots a_s}}{\partial \theta_1^\alpha} + \frac{1}{\mu} \frac{\partial U_{a_1 \dots a_s}}{\partial \theta_1^\alpha} F_{a_1 \dots a_s} \tag{23}$$

$$+ \frac{1}{4\pi\mu} \int \sum_{\substack{1 \leqslant a_{s+1} \leqslant N \\ (a_{s+1} \neq a_1, \dots a_s)}} \frac{\partial \Phi_{a_1, a_{s+1}}}{\partial \theta_1^\alpha} F_{a_1 \dots a_{s+1}} d\Omega_{s+1} = 0,$$

where $U_{a_1 \dots a_s}$ is the potential energy of the system of s chosen dipoles. All the calculations in this case are entirely analogous to those used by Bogoliubov for the distribution function of the positions of molecules in a gas or liquid.[1]

We write the interaction potential in the form

$$\Phi_{a_1 a_2} / \mu = \gamma \psi_{a_1 a_2}, \tag{24}$$

and will consider the quantity $\gamma = p^2 / d^3 \mu$ (p is the dipole moment of the molecule; for simplicity we assume that the magnitudes of the dipole moments of all the molecules are identical) as a parameter which is characteristic for the system under consideration.

If at first we do not consider the effect of the external electric field, and assume the inequality $\gamma << 1$ to be satisfied, then the equation

$$\frac{\partial}{\partial \theta_1^\alpha} F_{a_1 \dots a_s} + \gamma \frac{\partial \overset{*}{U}_{a_1 \dots a_s}}{\partial \theta_1^\alpha} F_{a_1 \dots a_s} \tag{25}$$

$$+ \frac{\gamma}{4\pi} \int \sum_{1 \leqslant a_{s+1} \leqslant N} \frac{\partial \psi_{a_1, a_{s+1}}}{\partial \theta_1^\alpha} F_{a_1 \dots a_{s+1}} d\Omega_{s+1} = 0$$

$$U a_1 \dots a_s = \sum \varphi a_i a_k$$

can be solved with the help of an expansion in powers of the small parameter γ. Writing $F_{a_1 \dots a_s}$ in the form

$$F_{a_1 \dots a_s} = F^0_{a_1 \dots a_s} + \gamma F^1_{a_1 \dots a_s} + \dots, \tag{26}$$

we find

$$F^0_{a_1 \dots a_s} = \text{const.}$$

If the normalization condition is written in the form

$$(4\pi)^{-s} \int F^i_{a_1 \dots a_s} d\Omega_1 \dots d\Omega_s = 0, \tag{27}$$

$$i = 0, 1, \dots,$$

then $F^\circ_{a_1 \dots a_s} = 1$ follows from this equality.

The equation for $F^1_{a_1 \dots a_s}$ has the form

$$\frac{\partial}{\partial\theta_1^\alpha} F^1_{a_1\ldots a_s} + \frac{\partial U^*_{a_1\ldots a_s}}{\partial\theta_1^\alpha} + \frac{1}{4\pi}\int \sum_{1\leqslant a_{s+1}\leqslant N} \frac{\partial \psi_{a,a_{s+1}}}{\partial\theta_1^\alpha}\, d\Omega_{s+1} = 0, \tag{28}$$

and since, for the dipole potential,

$$\int \psi_{a_1,\, a_{s+1}} d\Omega_{s+1} = 0, \qquad \text{identically,}$$

we get

$$F^1_{a_1\ldots a_s} = -U^*_{a_1\ldots a_s} \tag{29}$$

in agreement with the normalization condition. The second approximation is found in a similar way and has the form

$$F^2_{a_1\ldots a_s} = \frac{1}{2}\Big(U^{*2}_{a_1\ldots a_s} + \frac{1}{4\pi}\sum_{a_{s+1}}^{N}\int\Big(\sum_{a_i}^{s}\psi_{a_i a_{s+1}}\Big)^2 d\Omega_{s+1}\Big) + K^{(2)}_{a_1\ldots a_s}\Big), \tag{30}$$

where, in accord with the normalization condition, the constant K must be given by

$$K^{(2)}_{ai\ldots a_s} = -\frac{1}{2}\frac{1}{(4\pi)^s}\Big\{\int \sum_{a_i a_k}^{s} \psi^2_{a_i a_k} d\Omega_1 \ldots d\Omega_s + \frac{1}{4\pi}\int \sum_{a_{s+1}}^{N}\sum_{a_i}^{s}\psi^2_{a_i a_{s+1}}\, d\Omega_1 \ldots d\Omega_{s+1}\Big\}. \tag{31}$$

Here we have studied the properties of the dipole potential ψ. In similar fashion the constants for further approximations can be determined, but their calculation is not so simple.

In the presence of an external homogeneous electric field E and upon fulfillment of the condition

$$\gamma = p^2/d^3\mu \ll 1 \tag{32}$$

the computation scheme does not undergo any change.

These expressions for the distribution function permit us to construct a theory of the equilibrium properties of the dipole crystals under consideration. Numerical calculations and comparison with experiment, and also comparison with other theories which apply to the two very simple cases considered by us will be given in a subsequent paper. Separate consideration is necessary for the construction of a theory in the case of violation of the relation (32), while this case has the greatest interest. We hope in the near future to publish results which apply to this variant.

For the second of the problems considered by us the considerations developed above can without essential change be applied for the purpose of constructing a semiclassical theory of ferromagnetic bodies without use of the "sphericalizing" approximation.

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Relaxation between Electrons and the Crystalline Lattice*

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The relaxation between the electrons of a metal and the crystalline lattice (phonons) is considered. The state of the electrons and the lattice is described by equilibrium Fermi and Bose functions with different temperatures. The heat transfer coefficient connected with the "Cerenkov" radiation of sound waves by the electrons has been determined.

1. THERE have appeared recently several experimental[1,2] and theoretical[3,5] papers illuminating the investigation of deviations from Ohm's law in metals. The departures from a linear relation between the current J and the field E, noted by

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Borovik,[1] are observed in the presence of a large current when the mean energy of the electrons differs appreciably from that which corresponds to the lattice temperature T. Since the time required to establish equilibrium in the electron gas is much less than the time for achieving equilibrium between the electrons and the lattice,[4] we can consider that the electron gas is in a state of equilibrium, i.e., its state is described by the ordinary Fermi distribution function

$$n = [1 + \exp(\varepsilon - \varepsilon_0)/k\Theta]^{-1}, \tag{1}$$

$$\varepsilon_0 = (3n_0/8\pi)^{2/3}(2\pi\hbar)^2/2m,$$

n_0 is the conduction electron density, m is the effective electronic mass. The electron temperature Θ here differs from the lattice temperature ($\Theta > T$).

A similar state of partial equilibrium can arise not only in the passage of electric current through the metal. For example, in the passage of fast charged particles through material, the energy of the particle is spent in direct interaction with the ions of the lattice (nuclear collisions) and by interaction with electrons (ionization, polarization). Here a major part of the energy of the particles is spent in interaction with the electrons (~ 90% for fission fragments, see Ref. 6). As a result, large local temperature differences arise between the electrons and the lattice. Subsequently, the transfer of energy from the electrons to the lattice takes place by means of a relaxation mechanism, described below.*

The temperature regimes of the electron gas (the maximum temperature difference between the electrons and the lattice, the relaxation time) are determined by the source of the heat and by the coefficient of heat transfer from the electrons to the lattice. At high temperatures ($T >> T_0$, T_0 the Debye temperature) the problem was solved by Ginzburg and Shabanskii.[4] In the present work a method is used which permits us to find the heat transfer coefficient for arbitrary temperatures.

2. Let us compute the amount of energy transferred by the electrons (per unit volume) to the lattice in unit time ($\overline{U}$). Evidently,**

$$\overline{U} = \int \dot{N}_f \hbar\omega_f (2\pi)^{-3} d\tau_f V. \tag{2}$$

Here N_f = change (per unit time) in the number of phonons with wave vector f and energy $\hbar\omega_f = \hbar s f$ (per unit volume), s = sound velocity, V = volume of the crystal. We designate the probability (per second) of transition of the electron from a state with wave vector k into a state with wave vector k$'$ by $W_{k,k'}\delta(\epsilon_k + \hbar\omega_f - \epsilon_{k'})$; the δ-function obeys the law of energy conservation*

$$W_{k,k'} = (\pi U^2/\rho V s^2)\omega_f; \qquad f = k - k'. \tag{3}$$

Then[8]

$$\dot{N}_f = \int W_{k,k'}\{(N_f + 1)n_{k'}(1 - n_k) - N_f n_k(1 - n_{k'})\} \times \delta(\varepsilon_k + \hbar\omega_f - \varepsilon_{k'})(2\pi)^{-3}\,2d\tau_{k'}. \tag{4}$$

As is seen from (3) and (4), the change in the phonon function takes place because of the "creation" and "annihilation" of the phonons. Similar processes can exist, since the velocities of the electrons are greater than the velocity of sound, which guarantees simultaneous fulfillment of the laws of conservation of momentum and energy. This shows that the creation of the phonon in the quantum-mechanical description corresponds to Cerenkov radiation of sound waves by electrons in the classical view (see below).

For identical temperatures of electrons and phonons (lattice) the expression under the integral vanishes identically for the equilibrium distribution function. In our case, when $T \neq \Theta$ and

$$N_f = [e^{\hbar\omega_f/kT} - 1]^{-1}, \tag{5}$$

we have

$$\dot{N}_f = \frac{m^2U^2\hbar\omega_f}{2\pi\hbar^4\rho V_S}\,\frac{e^{\hbar\omega/kT} - e^{\hbar\omega/k\Theta}}{(e^{\hbar\omega/kT} - 1)(e^{\hbar\omega/k\Theta} - 1)}. \tag{6}$$

For computation, we have considered that $\epsilon_k = \hbar^2 k^2/2m$ and that $\epsilon_0 >> ms^2/2$ and $k\Theta$. The latter conditions are always satisfied beforehand. For Bi, for which $\epsilon_0 \sim kT_0$, this limits the region of application of Eq. (6) to low temperatures.

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*A separate publication will be devoted to consideration of the interaction of nuclear radiation with matter.

** We use the method developed by Akhiezer and Pomeranchuk[7] for the calculation of the spin-lattice exchange energy.

*The following notation is used: ρ = density of the material ($\rho = M/d^3$, M is the mass of the lattice atom, V = lattice volume); U = constant interaction of the electron with the lattice that appears in the expression for the time to travel the mean free path (see, for example, Refs.8,9 where it is denoted by C).

Substituting (6) in (2) and integrating, we get

$$\overline{U}=\frac{2}{(2\pi)^3}\frac{m^2U^2(kT_0)^5}{\hbar^7\rho s^4} \tag{7}$$

$$\times\left\{\left(\frac{\Theta}{T_0}\right)^5\int\limits_0^{T_0/\Theta}\frac{x^4dx}{e^x-1}-\left(\frac{T}{T_0}\right)^5\int\limits_0^{T_0/T}\frac{x^4\,dx}{e^x-1}\right\}.$$

In the limiting case of low temperatures (T, $\Theta << T_0$), we have

$$\overline{U}=\frac{2D_5}{(2\pi)^3}\frac{m^2U^2(kT_0)^5}{\hbar^7\rho s^4}\frac{\Theta^5-T^5}{T_0^5}; \tag{8}$$

$$D_5=\int\limits_0^\infty\frac{x^4\,dx}{e^x-1};$$

in the opposite case, at high temperatures (T, $\Theta >> T_0$),

$$\overline{U}=\frac{m^2U^2(kT_0)^5}{2\hbar^7\rho s^4(2\pi)^3}\frac{\Theta-T}{T_0}. \tag{9}$$

Comparing Eqs. (8) and (9) with the expressions for the time of flight (Ref. 9, p. 184), we get for low temperatures,

$$\overline{U}=(2\pi^2/15)\,ms^2n_0\{1/\tau(\Theta)-1/\tau(T)\}, \tag{10}$$

and for high,

$$\overline{U}=(\pi^2/6)\,ms^2n_0\{1/\tau(\Theta)-1/\tau(T)\}. \tag{11}$$

Here, we understand by $\tau(\Theta)$ and $\tau(T)$ the time of free flight of electrons under the condition that the lattice temperature coincides with the electron temperature and is equal respectively to Θ or T; n_0 is the number of electrons per unit volume.

If the temperature difference $\Theta - T$ is much less than the lattice temperature T, then

$$\overline{U}=\frac{2\pi^2}{3}\frac{ms^2n_0}{\tau(T)}\frac{\Theta-T}{T} \tag{12}$$

$$(T\ll T_0;\ \Theta-T\ll T);$$

$$\overline{U}=\frac{\pi^2}{6}\frac{ms^2n_0}{\tau(T)}\frac{\Theta-T}{T} \tag{13}$$

$$(T\gg T_0;\ \Theta-T\ll T);$$

The latter expression differs from Eq. (4) of Ref. 4 by the appearance of the factor $\pi^2/6$ in place of the 2 in the earlier expression.

If the lattice temperature is much less than the temperature of the electrons (hardly a practical case), then

$$\overline{U}=\begin{cases}\dfrac{2\pi^2}{15}\dfrac{ms^2n_0}{\tau(\Theta)} & (\Theta\ll T_0;\ T\ll\Theta);\\[2ex] \dfrac{\pi^2}{6}\dfrac{ms^2n_0}{\tau(\Theta)} & (\Theta\gg T_0;\ T\ll\Theta).\end{cases} \tag{14}$$

3. Equation (13) can be obtained by purely classical means. Let us consider for this purpose the radiation of sound waves by an electron moving through the lattice with constant velocity v. Similar consideration was given by Buckingham [10] but the method used was excessively complicated.

The equations for excited vibrations of the elastic continuum have the form

$$\ddot{\mathbf{u}}_l-s^2\Delta\mathbf{u}_l=-(U/\rho)\nabla\delta(\mathbf{r}-\mathbf{v}t), \tag{15}$$

$\mathbf{u}_l$ is the longitudinal component of the displacement vector; the transverse component is equal to zero identically, since the actual force on the part of the electron is "longitudinal" [curl F= 0; F= $-(U/\rho)\nabla\delta(r-vt)$, the δ-character of the force shows that the electron, as a consequence of the screening action of the atom, lies within the limits of single cell].

The energy losses of the electron can be computed as the work done by the force exerted by the electron on the medium (cf. with Ref. 11),

$$\frac{d\varepsilon}{dt}=U\int\dot{\mathbf{u}}_l\nabla\delta(\mathbf{r}-\mathbf{v}t)\,dV. \tag{16}$$

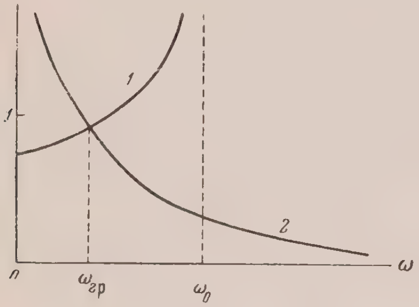
Expanding the δ-function in a Fourier integral, we determine the Fourier components of the displacement vector from (15):

$$\mathbf{u}_\mathbf{k}=\frac{iU}{\rho\,(2\pi)^3}\,\mathbf{k}\,\frac{e^{-i(\omega t-\mathbf{kr})}}{\omega^2-s^2k^2};\quad\omega=vk. \tag{17}$$

Hence

$$\mathbf{u}=\frac{iU}{\rho\,(2\pi)^3}\int\frac{\mathbf{k}e^{-i(\omega t-\mathbf{kr})}}{\omega^2-s^2k^2}\,d\tau_\mathbf{k}. \tag{18}$$

Differentiating (18) with respect to the time and substituting in Eq. (16), making use of the properties electric field E. Step-by-step solution of the problem reduces to the solution of a system of kinetic equations for the electronic and phonon distribution functions, in analogy with the known work of Davydov on semiconductors.[14] A similar calculation for metals has been given recently.[5] However the method, applied by the author, meets with several objections although the results obtained in Ref. 5 are evidently correct (see below) for not too high fields, i.e., for small differences of the electron and phonon temperatures.



Curve $1 - \frac{1}{1-\omega/\omega_0} - \frac{s_0^2}{v^2}$ as a function of ω, curve $2 - \varkappa_m^2 s_0^2/\omega^2$ as a function of ω

If the difference of temperatures $\Theta - T$ is small in comparison with the lattice temperature T, then for the determination of the temperature of the electrons it is necessary to make use of the heat balance equation

$$c\dot{\Theta} = -\overline{U} + q \tag{26}$$

[$c = (\pi/2)^2 kn_0 kT/\epsilon_0$ is the electronic specific heat, q is the strength of the heat source]. In this case one can regard the phonon temperature as given, since the specific heat of the lattice down to very low temperatures ($\lesssim$ 1° K) is appreciably larger than the specific heat of the electrons. In our case, $q = \sigma E^2 = j^2/\sigma$, where $\sigma = ne^2\tau/m$ is the real conductivity of the metal. Within the limits of applicability of Matthiessen's rule[9]

$$1/\tau = 1/\tau(T) + 1/\tau_{\text{imp}}, \tag{27}$$

where τ_{imp} is time of free flight relative to the impurity. From Eq. (26), making use of Eqs. (12) and (13), we find the equilibrium temperature of the electrons (for which $\dot{\Theta} = 0$):

$$\frac{k(\Theta - T)}{\varepsilon_0} = \frac{3}{4\pi^2}\left(\frac{elE}{\varepsilon_0}\right)^2 \frac{l}{L}\frac{kT}{ms^2} \qquad (T \ll T_0); \tag{28}$$

$$\frac{k(\Theta - T)}{\varepsilon_0} = \frac{6}{\pi^2}\left(\frac{elE}{\varepsilon_0}\right)^2 \frac{kT}{ms^2} \qquad (T \gg T_0). \tag{29}$$

Here $l = \tau(T)v_0$, $L = \tau v_0$. The expressions that have been obtained coincide, with accuracy to within a numerical factor, with the expressions found in Ref. 5 [Eq. (12)] and in Ref. 4.

Equation (28) cannot be regarded as correct down to absolute zero. It is evident from (14) that the heat emission will take place even for $T = 0$. This leads to the following value of the equilibrium electron temperature:

$$\Theta_{\text{p}} = T_0\left\{\frac{15}{2\pi^2}\frac{(elE)^2}{\varepsilon_0^2}\frac{\varepsilon_0}{ms^2}\frac{l_0}{L}\right\}^{1/5}. \tag{30}$$

By l_0 is meant the value of the length of mean free path at the Debye temperature.*

Making use of Eq. (26), it is easy to find the relaxation time t_0 (it is identical with the time required to establish temperature equilibrium)

$$t_0 = \alpha\tau(T)(kT)^2/ms^2\varepsilon_0. \tag{31}$$

Here $\alpha = 3/4$ for $T \ll T_0$ and $\alpha = 3$ for $T \gg T_0$. Since $\epsilon_0/k \sim 10^4$ °, $v_0 \sim 10^3\, s$, then t_0 becomes of the same order as $\tau(T)$ for $T \sim 10$° K. It is evident that our consideration would be invalid below this temperature if the metal did not contain impurities. However, for low temperatures, $\tau(T) \gg \tau$ practically always, so that even in this case the relaxation time (time for the transfer of energy from the electrons to the lattice) is significantly greater than the time for establishing the temperature inside the electron gas (for details, see Ref. 4).

If we consider that the lattice is maintained all the time at very low temperature ($T = 0$), then the established temperature is determined by Eq. (30) while the process of equalizing the temperature takes place according to the following law:

$$\Theta = \Theta_0(1 + t/t_0)^{-1/3},$$
$$t_0 \sim (kT_0)^2(ms^2\varepsilon_0)^{-3/5}\tau_0(L/l_0)^{3/5}.$$

The effect of the departure of the electron temperature from the phonon temperature on the electrical of the δ-function, we find

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*This is a purely formal definition: $\tau(T_0) = (T/T_0)^5 \times \tau(T)$. By $\tau(T)$ is understood the expression for the time of free flight, which is valid for temperatures considerably less than the Debye temperature.

$$\frac{d\varepsilon}{dt} = -\frac{iU^2}{(2\pi)^3\rho}\int\frac{k^2\omega\, d\tau_{\mathbf{k}}}{\omega^2 - s^2k^2}. \qquad (19)$$

The integration is carried out over all k-space. In order that the integral have meaning, it is necessary to consider that s has a small imaginary addition (which later is made to approach zero) which corresponds to sound absorption in the medium. Here $s(-\omega) = s^*(\omega)$. With the help of the latter equality, Eq. (19) can be written as

$$\frac{d\varepsilon}{dt} = -\frac{U^2}{(2\pi)^2\rho v} \qquad (20)$$

$$\times\, 2\mathrm{Re}\, i\int_0^\infty\int_0^{\varkappa_m}\frac{(\varkappa^2 + \omega^2/v^2)\,\varkappa\, d\varkappa\, \omega\, d\omega}{\omega^2(1 - s^2/v^2) - s^2\varkappa^2}.$$

We transformed to a cylindrical coordinate system with its axis along the vector **v**, $\varkappa_m$ is the maximum value of the component $\varkappa$ of the wave vector **k**. We integrated over $\varkappa$ up to the upper limit $\varkappa_m$ keeping in mind the fact that for very short waves (over very short distances), the crystal cannot be regarded as continuous. Evidently $\varkappa_m$ must be of the order of π/d (d= lattice constant). For a more accurate choice of the value of $\varkappa_m$ see below.

Integrating first over $\varkappa$, and noting that the component which does not contain the logarithm is known to be equal to zero, we get

$$\frac{d\varepsilon}{dt} = \frac{U^2}{(2\pi)^2\rho v s^4} \qquad (21)$$

$$\times\, \mathrm{Re}\, i\int_0^\infty \omega^3\, d\omega\, \ln\left[1 + \frac{s^2\varkappa_m^2}{\omega^2[(s^2/v^2) - 1]}\right].$$

It is evident from Eq. (21) that the real contribution to the loss is made by those frequencies for which

$$1 + \frac{s^2\varkappa_m^2}{\omega^2[(s^2/v^2) - 1]} < 0. \qquad (22)$$

Then

$$\frac{d\varepsilon}{dt} = \begin{cases} 0 & (v < s), \\ -\dfrac{1}{16\pi}\dfrac{U^2\varkappa_m^4}{\rho v(1 - s^2/v^2)^2} & (v > s). \end{cases} \qquad (23)$$

The latter expression coincides with the expression obtained in Ref. 10.

The divergence which takes place for v=s (and, it would seem, which has no analog in Cerenkov radiation of electromagnetic waves), is evidently connected with the fact that we have not taken dispersion into account (the dependence of the sound velocity on the frequency). Actually, it is evident from Eq. (22) that the divergence at v=s is brought about by the fact that the limiting frequency becomes infinite (i.e., the frequency below which Eq. (22) is satisfied). Account of dispersion leads to finiteness of the limiting frequency for all values of the particle velocity.* For example, if we choose the dependence of the sound velocity on the frequency in the form

$$s = s_0(1 - \omega/2\omega_0), \qquad (24)$$

which corresponds to account of quadratic terms in the dependence of the frequency ω on the wave vector f, then the limiting frequency ω_{lim} is determined by the equation and is finite for all values of

$$\frac{1}{1 - \omega_{\mathrm{rp}}/\omega_0} - \frac{s_0^2}{v^2} = \frac{\varkappa_m^2 s_0^2}{\omega_{\mathrm{rp}}^2}$$

the electron's velocity (see Figure). We note that, with account of dispersion, there is always a region in which condition (22) is satisfied, i.e., there is a region of Cerenkov radiation of sound for any velocity. This is connected with the fact that, according to the assumption of Eq. (24), the velocity of sound for high frequencies ($\omega \sim \omega_0$) falls to zero (cf. Ref. 13).

It is evident that in the transfer of energy from the electron to the lattice, only those electrons take part which are located close to the Fermi surface,[4] i.e.,

$$\overline{U} = \frac{1}{16\pi}\frac{U^2\varkappa_m^4 n_0}{\rho v_0}\frac{(\Theta - T)\,k}{\varepsilon_0}. \qquad (25)$$

We omit the factor $(1 - s^2/v_0^2)^2$, since $s \ll v_0$ (v_0 is the limiting velocity of the electrons). We choose $\varkappa_m$ so that Eq. (25) coincides with Eq.(9). Making use of the expression for ϵ_0 and $T_0 = \hbar\pi s/dk$ we obtain

$$\varkappa_m = (3\pi^2)^{1/4}\pi/d,$$

which agrees in order of magnitude with the limiting wave vector π/d.

4. Up to now we have considered the temperature of the electrons as given. For its computation it is necessary to give a mechanism of heating the electrons. Let us consider the effect of a strong

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*We recall that without account of dispersion, the energy losses of the electron in Cerenkov radiation also become infinite,[12] and for the same reason.

conductivity and the thermoelectric emission is considered in further detail in Ref. 4.

In conclusion the authors take this occasion to express their gratitude to E. S. Borovik for discussions on the problem.

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Determination of the Green's Function in the Bloch-Nordsieck Model by Functional Integration

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In an electrodynamic model where the Dirac γ matrices are replaced by c-numbers, there is no vacuum polarization and the equations for the Green's function of an electron in an external field can be solved by quadratures. The Green's function with radiative corrections is then easily obtained by functional integration. The expression obtained for the Green's function after renormalization is manifestly analytic and has no infinities.

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1. INTRODUCTION

SEVERAL papers[1-8] have recently appeared on functional integration as applied to quantum field theory. This method is of interest from two points of view: first, the mathematical apparatus seems the most adequate way of describing a field as a system with an infinite number of degrees of freedom, so that within the mathematical scheme one can obtain solutions of the field equations in closed form: while secondly the functional integration method does not assume the coupling constant to be small, so that one may make other approximations than those founded on perturbation theory.

However, although it has been possible to express the fundamental quantities in the theory as functional integrals, the method has not been able to produce concrete physical results because so far only the gaussian functional could be integrated.

From a methodological point of view, it is interesting at this stage to consider several simple models where the integration can be carried out. In Refs. 1 and 8, functional integration methods were applied to the interaction between an infinitely heavy nucleon and a scalar meson field, Ref. 1 considering a neutral and Ref. 8 a charged field.

In the present paper, we consider, as an example where the calculations can be carried out in full, the the Bloch-Nordsieck model in electrodynamics. As is well known, Bloch and Nordsieck proposed to avoid the infrared catastrophe in the interaction of particles with light at low frequencies by an approximate method not based on perturbation theory. It turned out that their zero order approximation was equivalent to replacing the Dirac γ matrices by c-numbers u_α

$$u^2 = g^{\alpha\alpha} u_\alpha^2 = u_0^2 - \mathbf{u}^2 = 1. \qquad (1)$$

Here

$$g^{\alpha\alpha} = \begin{cases} 1 & \text{for } \alpha = 0, \\ -1 & \text{for } \alpha = 1, 2, 3. \end{cases} \qquad (2)$$

We shall look at this model from the Green's function point of view, and start with the representation, obtained in Ref. 3, for the Green's function of an electron as a functional integral depending on the Green's function of an electron in a classical external field. The integral will be calculated and then renormalized in the following sections.

First note that there is no vacuum polarization in this model, so that we may immediately assume

$$\langle S \rangle_{F_0} \qquad (3)$$
$$= \exp\left\{ \sqrt{4\pi}\, e \int_0^1 d\lambda \int dx\, G(x, x \mid \lambda A)\, uA(x) \right\} = 1.$$

Formula (3) is the well-known expression for the average of the S-matrix in a Fermi vacuum (see, for example Ref. 3); we use the Feynman notation for the scalar product of four-vectors:

$$ab = g^{\alpha\alpha} a_\alpha b_\alpha = a_0 b_0 - \mathbf{a} \cdot \mathbf{b}.$$

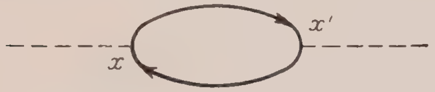
That there is no vacuum polarization is clear from the following considerations. The Green's function $S^c\ (x - x')$ for a free electron is determined in the present case by the first order equation

$$[i u_\alpha (\partial / \partial x_\alpha) - m]\, S^c (x - x') = -\delta (x - x') \qquad (4)$$

and has one pole, not two as in ordinary electrodynamics:

$$S^c (x - x') = \frac{1}{(2\pi)^4} \int \frac{e^{-ip(x-x')}}{m - up - i\varepsilon}\, dp. \qquad (5)$$

Hence $S^c(x-x')=0$ for $t'>t$, i.e., this is a retarded Green's function. Then, in the matrix element corresponding to a diagram of the type



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one of the electron lines will correspond to $S^c \times (x-x')=0$. Similarly, all more complicated diagrams corresponding to vacuum polarization will be zero. Physically, the absence of a second pole in the Green's function means that there are no antiparticles in the theory, and hence that pairs cannot be created. Since the contribution of closed loops is zero, the photon Green's function in this model is identical with its zero order approximation in the perturbation theory.

2. GREEN'S FUNCTION FOR AN ELECTRON IN AN EXTERNAL FIELD

In the present case, the Green's function for an electron in a given external field is determined by the equation

$$[iu_\alpha(\partial/\partial x_\alpha)-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m] \times G(x,x'\,|\,A)=-\delta(x-x'), \qquad (6)$$

which can be solved by quadratures. To do this, we use Fock's proper time method [10], which involves the formula

$$H^{-1}=-i\int_0^\infty d\nu e^{iH\nu-\varepsilon\nu}. \qquad (7)$$

For our special case

$$H=iu_\alpha(\partial/\partial x_\alpha)-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m, \qquad (8)$$

so that the solution of (8) can be written symbolically

$$G(x,x'\,|\,A)=i\int_0^\infty d\nu \exp\left\{i\nu\left(iu_\alpha\frac{\partial}{\partial x_\alpha}-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m+i\varepsilon\right)\right\}\delta(x-x'). \qquad (9)$$

Introducing a new unknown function

$$U(\nu)=\exp\left\{i\nu\left(iu_\alpha\frac{\partial}{\partial x_\alpha}-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m+i\varepsilon\right)\right\}\delta(x-x'), \qquad (10)$$

we find that it satisfies the differential equation

$$-i\frac{\partial U(\nu)}{\partial\nu}=\left(iu_\alpha\frac{\partial}{\partial x_\alpha}-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m+i\varepsilon\right)U(\nu) \qquad (11)$$

with the initial condition with respect to the proper time parameter ν

$$U(0)=\delta(x-x'). \qquad (12)$$

Taking the Fourier transform of the δ function

$$\delta(x-x')=(2\pi)^{-4}\int e^{-ip(x-x')}\,dp \qquad (13)$$

we seek a solution of (11) in the form

$$U(\nu)=(2\pi)^{-4}\int \exp\{S(p,\nu)\}\,dp. \qquad (14)$$

The function $S(p,\nu)$ satisfies the equation

$$-i\frac{\partial S}{\partial\nu}=iu_\alpha\frac{\partial S}{\partial x_\alpha}-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x)-m+i\varepsilon \qquad (15)$$

with the initial condition

$$S(\nu;p)\,|_{\nu=0}=-ip(x-x'). \qquad (16)$$

Finally, the substitution

$$S=-ip(x-x')-i(m-up-i\varepsilon)\nu+R, \qquad (17)$$

leads to the following equation for R

$$-i\frac{\partial R}{\partial\nu}=iu_\alpha\frac{\partial R}{\partial x_\alpha}-\sqrt{4\pi}eg^{\alpha\alpha}u_\alpha A_\alpha(x) \qquad (18)$$

and the initial condition

$$R(0)=0. \qquad (19)$$

Equation (18), with the initial condition (19), is easily solved by Fourier transforms. The result is

$$R = -i\frac{\sqrt{4\pi}\,e}{(2\pi)^4}\int dku A(k)\,e^{-ikx}\int_0^{\nu} d\nu' e^{iuk\nu'}, \tag{20}$$

$$G(x, x' \mid A) = i\int_0^{\infty} d\nu$$
$$\exp\{-ip(x-x') - i(m-up-i\varepsilon)\nu + R(\nu \mid A)\}. \tag{21}$$

3. THE ELECTRON GREEN'S FUNCTION WITH RADIATIVE CORRECTIONS

As was shown in Ref. 3, the Green's function with radiative corrections can be written as the ratio of two functional integrals:

$$G(x-x') = \frac{\int G(x, x' \mid A)\exp\{i\int L_f^0\,dx\} < S >_{F_0} \delta A}{\int \exp\{i\int L_f^0 dx\} < S >_{F_0} \delta A}. \tag{22}$$

Here $\int L_0^f \ dx$ is the action function for an electromagnetic field and can be written

$$\int L_0^f dx = \frac{g^{\alpha\alpha}}{2(2\pi)^4}\int A_\alpha(k)\,D^{c^{-1}}(k, k')\,A_\alpha(k')\,dkdk', \tag{23}$$

where

$$D^c(k, k') = D^c(k)\,\delta(k+k'),$$
$$D^c(k) = -1/(k^2+i\varepsilon). \tag{24}$$

In our case we have the important simpification $< S >_{F_0} = 1.$

Hence we obtain

$$G(x-x') = c^{-1}\int G(x, x' \mid A)\exp\left\{i\int L_f^0\,dx\right\}\delta A, \tag{25}$$

where the denominator of formula (22), which is a constant, has been denoted by c . From the form of $G(x,x' \mid A)$ it is clear that in order to calculate (25) it is sufficient to find

$$\int \exp\left\{R(\nu \mid A)\right\} e^{i\int L_f^0 dx}\,\delta A \tag{26}$$
$$= \int \exp\left\{-\frac{i}{(2\pi)^4}\int g^{\alpha\alpha}F_\alpha(k)\,A_\alpha(k)\,dk\right.$$
$$\left.+\frac{i}{2(2\pi)^4}\int g^{\alpha\alpha}A_\alpha(k)\,D^{c^{-1}}(k, k')\,A_\alpha(k')\,dk\,dk'\right\}\delta A,$$

where for convenience we have written

$$F_\alpha(k) = \sqrt{4\pi}\,eu_\alpha e^{-ikx}\int_0^{\nu} d\nu' e^{iuk\nu'}; \tag{27}$$

The functional integral (25) is of the gaussian type and can easily be calculated with the help of the transformation

$$A_\alpha(k) = A'_\alpha(k) + \int F_\alpha(k')\,D^c(k, k')\,dk'. \tag{28}$$

With this substitution the integral takes the form

$$\int \exp\left\{R(\nu \mid A)\right\}\exp\left\{i\int L_f^0\,dx\right\}\delta A \tag{29}$$
$$= c\exp\left\{-\frac{i}{2(2\pi)^4}\int g^{\alpha\alpha}F_\alpha(k)\,D^c(k, k')\,F_\alpha(k')\,dkdk'\right\}.$$

Then, using (25) and (27), we obtain

$$\int \exp\left\{R(\nu \mid A)\right\}\exp\left\{i\int L_f^0\,dx\right\}\delta A \tag{30}$$
$$= c\exp\left\{-\frac{ie^2}{2(2\pi)^3}\int dkD^c(k)\int_0^{\nu} d\nu_1 e^{-iuk\nu_1}\int_0^{\nu} d\nu_2 e^{-iuk\nu_2}\right\}.$$

Noting now that the infinite constant c cancels in (25), we finally obtain the following expression for the Green's function

$$G(x-x') = i\int_0^{\infty} d\nu\exp\{-ip(x-x')$$
$$-i(m-up-i\varepsilon)\nu + f(\nu)\}, \tag{31}$$

where

$$f(\nu) = \exp\left\{-\frac{ie^2}{2(2\pi)^3}\int dkD^c(k)\int_0^{\nu} d\nu_1 e^{-iuk\nu_1}\int_0^{\nu} d\nu_2\,e^{iuk\nu_2}\right\}. \tag{32}$$

Since $D^c\ (k)$ is even with respect to k , the last formula, (32), can be put in the form

$$f(\nu) = \exp\left\{-\frac{ie^2}{(2\pi)^3}\int dkD^c(k)\int_0^{\nu} d\nu_1\int_0^{\nu_1} d\nu_2 e^{iuk\nu_2}\right\}, \tag{33}$$

In momentum space, the Green's function becomes

$$G(p) = i\int_0^\infty d\nu \exp\{i(up - m + i\varepsilon)\nu + f(\nu)\}. \tag{34}$$

Taking, then, $u = p/|p|$, the Green's function is

$$G(p) = i\int_0^\infty d\nu \exp\{i(|p| - m + i\varepsilon)\nu + f(\nu)\}. \tag{35}$$

4. RENORMALIZATION OF THE GREEN'S FUNCTION

There are infinities in the expression we have obtained for the Green's function. We use the Pauli-Villars regularization method [11], with one auxiliary mass M, to renormalize. To deal with the infrared catastrophe we introduce also the photon "mass" λ. Then we adopt the photon Green's function

$$D^c(k) = (\lambda^2 - k^2 - i\varepsilon)^{-1} - (M^2 - k^2 - i\varepsilon)^{-1}. \tag{36}$$

At the end of the calculation we must go to the limits $M \to \infty$ and $\lambda \to 0$. Hence in carrying out the integration (33), it is sufficient to obtain asymptotic values for $\lambda \sim 0$.

Carrying out the integration on ν_1 and ν_2, we obtain for $f(\nu)$ the expression

$$\begin{aligned} f(\nu) &= -\frac{ie^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{1}{(uk)^2} \\ &+ \nu\frac{e^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{1}{uk} \\ &+ \frac{ie^2}{(2\pi)^3}\int dk\left(\frac{1}{\lambda^2 - k^2 - i\varepsilon} - \frac{1}{M^2 - k^2 - i\varepsilon}\right)\frac{e^{iuk\nu}}{(uk)^2}. \end{aligned} \tag{37}$$

The last integral remains finite as $M \to \infty$, as does the second for $\lambda \to 0$, so in these last two we can go to the indicated limits immediately. All the integrals can be calculated easily using the formula

$$x^{-1} = -i\int_0^\infty e^{ix\alpha - \varepsilon\alpha} d\alpha \tag{38}$$

and integrating first over k, then over the auxiliary variable α. The result is

$$f(\nu) = -(ie^2/2)M\nu + (e^2/\pi)\ln(M/\lambda) + \frac{e^2}{\pi}\ln(\nu\lambda). \tag{39}$$

We note that λ cancels out and the result is finite at $\lambda = 0$.

The first term, proportional to $i\nu$, gives the radiative correction to the mass of the electron, and diverges linearly as $M \to \infty$. It can be included in the effective mass of the electron, the renormalized mass being

$$m_1 = m + (e^2/2\pi)M. \tag{40}$$

We write that part, $\tilde{f}(\nu)$, of $f(\nu)$ which remains after mass renormalization in the form

$$\tilde{f}(\nu) = (e^2/\pi)\ln(M/m_1) + (e^2/\pi)\ln(m_1\nu). \tag{41}$$

The term independent of ν diverges logarithmically as $M \to \infty$, and determines the renormalization constant for the Green's function

$$Z = (M/m_1)^{e^2/\pi}. \tag{42}$$

Hence renormalization has removed all infinities from the theory and we obtain the following renormalized Green's function $G_1(p)$:

$$G_1(p) = Z^{-1}G(p), \tag{43}$$

$$G_1(p) = i(m_1)^{e^2/\pi}\int_0^\infty d\nu \exp\{i(|p| - m_1 + i\varepsilon)\nu\}\,\nu^{e^2/\pi}. \tag{44}$$

To evaluate the integral we make the substitution

$$x = |m_1 - |p||\,\nu, \tag{45}$$

Neglecting terms of order e^2/π, we finally obtain

$$G_1(p) = (m_1 - |p|)^{-1}|1 - |p|/m_1|^{-e^2/\pi}. \tag{46*}$$

In conclusion, I should like to express my deep gratitude to N. N. Bogoliubov, under whose guidance the present work was done.

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*Selection of the coupling $D^c_{\alpha\beta}$ in transverse form would have led to a factor 3/2 in the exponent of (46). This can be shown with the aid of a gauge transformation.

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A Method of Obtaining Nonstationary Solutions of Boltzmann's Kinetic Equation

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A method is considered for obtaining nonstationary solutions of Boltzmann's kinetic equation. This method is free from the limitations of the Chapman-Enskog method. By way of an example, the dispersion of plane sound waves in a monatomic gas is considered.

1. IN recent times the most widely used method of solution of the Boltzmann equation has been the method of Chapman-Enskog[1]. However, this method is not applicable for a series of problems. Let us consider, for example, one such problem -- the problem of the dispersion of plane sound waves in a monatomic gas without consideration of the internal degrees of freedom of the atoms. Let the frequency of vibration of the external source be sufficiently low in comparison with the "characteristic frequency" of the gas, i.e., in comparison with the mean frequency of atomic collisions. Such a problem can be solved by making use of the equations of Navier-Stokes and Burnett[2], i.e., by making use of different approximations than those of the Chapman-Enskog method. If the frequency of vibration of the external source is comparable with or larger than the "characteristic frequency" of the gas, then these approximations lose their meaning. Actually, we use as the small parameter of the method of successive approximation employed in obtaining the equations of Navier-Stokes, Burnett, the ratio $\Delta t_p/\Delta t$, where Δt is a characteristic time interval for the process under consideration, for example, the period of vibration, Δt_p is the relaxation time of the gas. Thus we have as the condition of applicability of the Chapman-Enskog method the relation

$$\Delta t_p / \Delta t \ll 1. \qquad (1)$$

Another method is necessary, consequently, to obtain a solution differing widely from the quasi-equilibrium solution of the Boltzmann equation. Such a possibility is given by the method of "moments" (see, for example, Ref. 3).

Let us formulate a modification of this method. We use as the zeroth approximation the stationary solution of the Boltzmann equation. Then the first approximation gives the deviation of the density, velocity and temperature from the stationary distribution, and also the corresponding viscous force and heat flow. As conditions for the application of this method of small perturbations we have the relations

$$\frac{\Delta\rho}{\rho_0};\quad \frac{\Delta u_i}{c_0};\quad \frac{\Delta\Theta}{\rho_0 c_0^2};\quad \frac{p_{ij}}{\rho_0 c_0^2};\quad \frac{S_i}{\rho_0 c_0^3} \ll 1, \qquad (2)$$

where ρ_0, $\Theta_0 = kT_0 = mc_0^{\,2}$ are, respectively, the particle density and temperature in the stationary state; m is the mass of an atom of the gas; k is the Boltzmann constant; $\Delta\rho$, Δu_i, $\Delta\Theta$, p_{ij}, S_i are the deviations of the density, velocity and temperature from the stationary state and also the viscosity tensor and the heat flow (first approximation).

Thus we have the kinetic equation of Boltzmann:

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} = F_{\mathrm{B}} + \frac{1}{m}\frac{\partial f}{\partial v_i}\frac{\partial V}{\partial x_i}, \qquad (3)$$

where F_B is the Boltzmann collision integral, V is the potential of the external field. We set

$$f(\mathbf{r}, \mathbf{v}, t) = f_0(\mathbf{r}, \mathbf{v}) + f_1(\mathbf{r}, \mathbf{v}, t) + \ldots,$$

$$V(\mathbf{r}, t) = V_0(\mathbf{r}) + V_1(\mathbf{r}, t) + \ldots. \qquad (4)$$

Then, substituting (4) in (3), we get a system of equations of successive approximations of the described method:

$$v_i \frac{\partial f_0}{\partial x_i} = F_{\mathrm{B}}(f_0) + \frac{1}{m}\frac{\partial f_0}{\partial v_i}\frac{\partial V_0}{\partial x_i}; \qquad (5)$$

$$\frac{\partial f_1}{\partial t} + v_i \frac{\partial f_1}{\partial x_i} = F_{\mathrm{B}}(f_0, f_1) + \frac{1}{m}\frac{\partial f_1}{\partial v_i}\frac{\partial V_0}{\partial x_i} + \frac{1}{m}\frac{\partial f_0}{\partial v_i}\frac{\partial V_1}{\partial x_i}.$$

The normalization conditions for the function f are written in the following way:

$$\rho = \int f d\mathbf{v}; \qquad (6)$$

$$u_i = \frac{1}{\rho}\int v_i f d\mathbf{v}; \quad \Theta + \frac{mu^2}{2} = \frac{1}{\rho}\int \frac{mv^2}{2} f d\mathbf{v};$$

$$\Pi_{ij} = \frac{1}{\rho}\int m v_i v_j f d\mathbf{v}; \qquad S_i = \frac{1}{\rho}\int v_i \frac{mv^2}{2} f d\mathbf{v}.$$

From Eqs. (4) and (6) there stems the possibility of the following normalization for the functions of the different approximations:

$$\rho_0(\mathbf{r}) = \int f_0 d\mathbf{v}; \quad \Delta\rho(\mathbf{r}, t) = \int f_1 d\mathbf{v}; \ldots \qquad (7)$$

$$\rho_0 u_{0i} = \int v_i f_0 d\mathbf{v}; \quad u_{0i}\Delta\rho + \rho_0 \Delta u_i = \int v_i f_1 d\mathbf{v}, \ldots$$

with similar relations for Θ, Π_{ij}, S_i.

In many problems (in particular, in the problem noted above on the propagation of sound waves) we can set $V_0 \equiv 0$. Then the equation of zeroth approximation (5), with account of (7), has the well-known solutions of Maxwell:

$$f_0 = \rho_0 c_0^{-3} (2\pi)^{-3/2} \exp\{-v^2/2c_0^2\}, \qquad (8)$$

where ρ_0, c_0 are constants, $u_{0i} \equiv 0$.

We now write the equation of the first approximation of (5), introducing dimensionless functions and variables:

$$g_0(\boldsymbol{\xi}) = (2\pi)^{-3/2} e^{-\xi^2/2}; \quad f_0(\mathbf{v}) = \rho_0 c_0^{-3} g_0(\boldsymbol{\xi});$$
$$f_1(\mathbf{r}, \mathbf{v}, t) = \rho_0 c_0^{-3} g_1(\mathbf{r}, \boldsymbol{\xi}, t); \quad \boldsymbol{\xi} = \mathbf{v}/c_0. \qquad (9)$$

We get*

$$\frac{\partial g_1}{\partial t} + c_0 \xi_i \frac{\partial g_1}{\partial x_i} = F_{\mathrm{B}}(g_0, g_1) + \frac{1}{mc_0}\frac{\partial g_0}{\partial \xi_i}\frac{\partial V_1}{\partial x_i}. \qquad (10)$$

We shall seek the solution of (10) in the form of an expansion in Hermite polynomials:

$$g_1 = g_0 \sum_{n=0}^{\infty} \sum_{i, j, \ldots} \frac{1}{n!}\alpha^{(n)}_{i, j, \ldots} H^{(n)}_{i, j, \ldots}, \qquad (11)$$

where the coefficients α are unknown functions, $\mathbf{r}$, t, $H^{(n)}_{i,j,\ldots}$ are the orthonormal Hermite polynomials, for example,

$$H^{(0)} = 1; \quad H^{(2)}_{i, j} = \xi_i \xi_j - \delta_{ij};$$
$$H^{(1)}_i = \xi_i; \quad H^{(3)}_{i, j, k} = \xi_i \xi_j \xi_k - \xi_i \delta_{jk} - \xi_j \delta_{ik} - \xi_k \delta_{ij}.$$

Because of the orthonormality of these polynomials, we have the relation

$$\frac{1}{n!}\int g_0 H^{(m)} H^{(n)} d\boldsymbol{\xi} = \delta_{mn}; \quad \alpha^{(m)} = \int H^{(m)} g_1 d\xi. \qquad (12)$$

Here the indices i, j, ... are omitted.

Substituting (11) in (10), we obtain a system of linear differential equations for the unknown functions $\alpha^{(m)}$. This same system can be obtained by multiplying Eq. (10) by $H^{(m)}$ and integrating over ξ. We then get

$$\int H^{(m)} \frac{\partial g_1}{\partial t} d\boldsymbol{\xi} + c_0 \int \xi_i H^{(m)} \frac{\partial g_1}{\partial x_i} d\boldsymbol{\xi} \qquad (13)$$
$$= \int H^{(m)} F_{\text{Б}}(g_0, g_1) d\boldsymbol{\xi}.$$

We get for the first term in Eq. (13) [keeping Eq. (12) in mind]

$$\int H^{(m)} \frac{\partial g_1}{\partial t} d\boldsymbol{\xi} = \frac{\partial \alpha^{(m)}}{\partial t}. \qquad (14)$$

To transform the second term in Eq. (13), we make use of relations of the form

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* In view of the fact that the term which takes into account the external field is not essential in what follows, it is omitted.

$$H_i^{(1)} H^{(m)} = H_i^{(m+1)} + \delta_i H^{(m-1)}. \qquad (15)$$

Using Eqs. (11), (12) and (15), we get

$$c_0 \frac{\partial}{\partial x_i} \int H_i^{(1)} H^{(m)} g_1 \, d\xi$$
$$= c_0 \frac{\partial}{\partial x_i} \{\alpha_i^{(m+1)} + \delta_i \alpha^{(m-1)}\}. \qquad (16)$$

We can write the right hand side of Eq. (13) in the form

$$\int H^{(m)}_{i,\,j,\,\dots} F_{\mathrm{B}}(g_0 g_1)\, d\xi = c_0, \rho_0 \sum_{n=2}^{\infty}$$
$$\sum_{i,\,j,\,\dots} \frac{1}{n!} \alpha^{(n)}_{i,\,j,\,\dots} \int d\xi g_0(\xi) H^{(m)}_{i,\,j,\,\dots}(\xi) \int d\xi_1 g_0(\xi_1) Y^{(n)}_{i,j,\dots}$$
$$Y^{(n)}_{i,\,j,\,\dots} = |\xi_1 - \xi| \int_0^{\infty} \varepsilon d\varepsilon \int_0^{2\pi} d\varphi \{H^{(n)}_{i,\,j,\,\dots}(\xi_1')$$
$$+ H^{(n)}_{i,\,j,\,\dots}(\xi') - H^{(n)}_{i,\,j,\,\dots}(\xi_1) - H^{(n)}_{i,\,j,\,\dots}(\xi)\}\, d\varphi \qquad (17)$$

if we apply Eq. (11) and the laws of conservation of energy and momentum. The integrals $Y^{(n)}_{i,j,\dots}$ were computed (see, for example, Ref. 3), whereupon we obtained

$$Y^{(2)}_{ij} = \frac{C}{c_0} \{w^2 \delta_{ij} - 3 w_i w_j\},$$
$$Y^{(3)}_{ijk} = \frac{C}{2c_0} \{w^2 [W_k \delta_{ij} + W_j \delta_{ik} + W_i \delta_{jk}]$$
$$-3 [w_i w_k W_j + w_i w_j W_k + w_j w_k W_i]\}, \qquad (18)$$

where

$$C = c_0 2\pi w \int_0^{\infty} \varepsilon d\varepsilon \sin^2\theta \cos^2\theta,$$
$$w = |\xi_1 - \xi|; \quad W = |\xi_1 + \xi|.$$

Further calculations are carried out only for Maxwell molecules with interaction action $\Phi(r) = br^{-4}$. Then

$$C = 0.6841 \sqrt{8b/m}. \qquad (19)$$

Further simple calculations lead to the following values of the integrals in Eq. (17):

$$\int d\xi_1 g_0(\xi_1) Y^{(2)}_{ij} = \frac{C}{c_0} \{\delta_{ij} \operatorname{Sp} H^{(2)}_{ii} - 3 H^{(2)}_{ij}\}; , \qquad (20)$$

$$\int d\xi_1 g_0(\xi_1) Y^{(3)}_{ijk}$$
$$= \frac{C}{2c_0} \{9 H^{(3)}_{ijk} - H^{(3)}_i \delta_{ik} - H^{(3)}_j \delta_{ik} - H^{(3)}_k \delta_{ij}\},$$

where, for example, $H_1^{(3)} = H_{111}^{(3)} + H_{122}^{(3)} + H_{133}^{(3)}$.

Now we can write down for the unknown functions $\alpha^{(m)}$ a set of differential equations which are equivalent to the Boltzmann equation of the first approximation (10). Actually, taking into account Eqs. (13), (14), (16), (17) and (20), we get, with accuracy to terms of fourth order:

$$\frac{\partial \alpha^{(m)}}{\partial t} + c_0 \frac{\partial}{\partial x_i} \{\alpha_i^{m+1} + \delta_i \alpha^{(m-1)}\}$$
$$= C\rho_0 \sum_{i,\,j,\,k,\,\dots} \Big\{\alpha^{(2)}_{ij} \frac{1}{2!} \int d\xi g_0 H^{(m)} [\delta_{ij} \operatorname{Sp} H^{(3)}_{ii}$$
$$- 3 H_{ij}] - \frac{\alpha^{(3)}_{ijk}}{2} \frac{1}{3!} \qquad (21)$$
$$\times \int d\xi g_0 H^{(m)} [9 H^{(3)}_{ijk} - H^{(3)}_i \delta_{jk} - H^{(3)}_j \delta_{ik} - H^{(3)}_k \delta_{ij}] \Big\}.$$

We describe these equations in more detail for the very simple case of one dimensional flow. Here, we choose as unknown functions five coefficients $\alpha^{(n)}$ from (11):

$$\alpha^{(0)},\ \alpha^{(1)}_1,\ \alpha^{(2)}_{11},\ \alpha^{(2)}_{22} = \alpha^{(2)}_{33},\ \alpha^{(3)}_{111} = \alpha^{(3)}_{122} = \alpha^{(3)}_{133}. \qquad (22)$$

The physical meaning of these functions is made clear from conditions (6) and (7), which, along with Eq. (12), give

$$\alpha^{(0)} = \Delta\rho / \rho_0, \ \alpha^{(1)}_i = \Delta u_i / c_0,$$
$$\operatorname{Sp} \alpha^{(2)}_{ii} = \Delta\Theta / \rho_0 c_0^2;$$
$${}^1/_2 (\alpha_{ij} + \alpha_{ji}) = p_{ij} / \rho_0 c_0^2;$$
$$\alpha^{(3)}_{iii} = 3 S_i / 5 \rho_0 c_0^3 - 3 \Delta\rho / \rho_0;$$
$$(\Pi_{ij} = p + p_{ij} + \dots;$$
$$S_i = S_i^0 + S_i + \dots; \ S_i^{(0)} = 0).$$

Keeping in mind (22) and (12), we get from (21) the following system:

$$\frac{\partial \alpha^{(0)}}{\partial t} + c_0 \frac{\partial \alpha^{(1)}}{\partial x} = 0; \qquad \frac{\partial \alpha^{(1)}}{\partial t} + c_0 \frac{\partial}{\partial x} (\alpha_{11} + \alpha^{(0)}) = 0;$$
$$\frac{\partial \alpha_{11}}{\partial t} + c_0 \frac{\partial}{\partial x} (2\alpha^{(1)} + \alpha^{(3)}) = 2C\rho_0 (\alpha_{22} - \alpha_{11});$$
$$\frac{\partial \alpha_{22}}{\partial t} + c_0 \frac{\partial}{\partial x} \frac{1}{3} \alpha^{(3)} = C\rho_0 (\alpha_{11} - \alpha_{22}); \qquad (23)$$
$$\frac{\partial \alpha^{(3)}}{\partial t} + c_0 \frac{\partial}{\partial x} \frac{3}{5} (3\alpha_{11} + 2\alpha_{22}) = -2C\rho_0 \alpha^{(3)}$$

by setting $H^{(m)}$ successively equal to

$$H^{(m)} = H^{(0)};\ H_1^{(1)};\ H_{11}^{(2)};$$
$$H_{22}^{(2)} + H_{33}^{(2)};\ H_{111}^{(3)} + H_{122}^{(3)} + H_{133}^{(3)}.$$

(The indices which are superfluous for the one dimensional case have been omitted.)

The system (23) determines the functions $\alpha^{(n)}$ if the corresponding initial and boundary conditions are given. A solution of the Boltzmann equation in the form (11) is found by the same approximation. This solution will no longer be limited by the condition (1). Thus the method considered here of small perturbations is free from the limitation of the Chapman and Enskog method.

2. As an example, let us consider the problem of the dispersion of plane sound waves in a monatomic gas without regard to internal degrees of freedom of the gas atoms. This problem has been considered by a number of authors (see Refs. 4-7). The starting point for these researches is the Burnett equation^2. Thus in these researches there were analyzed only the quasi-equilibrium solutions, the limitation of which follows from the requirement (1) of the method of Chapman-Enskog.

We consider this problem, not limiting ourselves to quasi-equilibrium solutions. For this purpose, we employ the system of equations (23), which is suitable for the one dimensional flow of gas. We write the corresponding system for the Fourier components of the unknown functions $\alpha^{(n)}$ from (23) as

$$U\alpha^{(0)} + \alpha^{(1)} = 0,\ U\alpha^{(1)} + \alpha_{11} + \alpha^{(0)} = 0; \qquad (24)$$
$$U\alpha_{11} + 2\alpha^{(1)} + \alpha^{(3)} = -i2U_0(\alpha_{22} - \alpha_{11});$$
$$U\alpha_{22} + {}^1/_3\,\alpha^{(3)} = -iU_0(\alpha_{11} - \alpha_{22});$$
$$U\alpha^{(3)} + {}^9/_5\,\alpha_{11} + {}^6/_5\,\alpha_{22} = i2U_0\,\alpha^{(3)}.$$

Here $U = \omega/kc_0$, $U_0 = \omega_0/kc_0$, $\omega_0 = c\rho_0$; $\alpha_{ij}^{(n)}$ denotes the Fourier components of the corresponding functions from (23), ω is the vibration frequency, k the wave vector.

Equating the determinant of this system to zero, we obtain the necessary dispersion equation

$$(U^5 - {}^{26}/_5\,U^3 + 3U) - iU_0(5U^4 - 16U^2 + 5)$$
$$- 2U_0^2U(3U^2 - 5) = 0. \qquad (25)$$

We now assume that ω is a given quantity and introduce the variable $\mu = \omega/\omega_0$.

The nontrivial solutions of this equation are

$$U = \pm\left[\frac{1}{2A}(-B \pm \sqrt{B^2 - 4AD})\right]^{1/2}.$$
$$A = \mu^2 - 6 - i5\mu,$$
$$B = 10 - {}^{26}/_5\,\mu^2 + i16\mu,\ D = 3\mu^2 - i5\mu. \qquad (26)$$

We write them in the form*

$$U_+ = \alpha_+ + i\beta_+,\quad U_- = \alpha_- + i\beta_-.$$

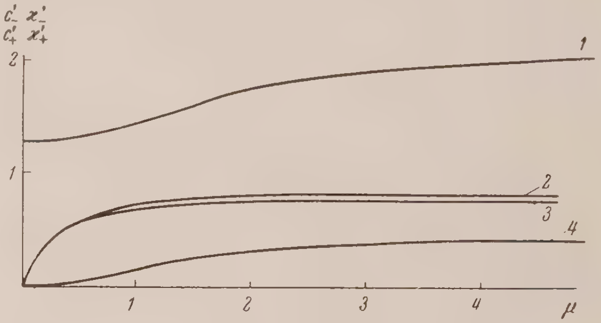
Introducing the phase velocity c and the absorption coefficient $\varkappa$ with the aid of the formulas

$$c = \omega/\operatorname{Re}k,\quad \varkappa = \operatorname{Im}k,$$

we finally obtain the solution of the dispersion equation for dimensionless velocity and absorption:

$$c'_\pm = c_\pm/c_0 = \varphi_1(\mu);\quad \varkappa'_\pm = \varkappa_\pm c_0/\omega_0 = \varphi_2(\mu),$$
$$\varphi_1(\mu) = [(\alpha_\pm)^2 + (\beta_\pm)^2]/\alpha_\pm;$$
$$\varphi_2(\mu) = \mu\beta_\pm/[(\alpha_\pm)^2 + (\beta_\pm)^2]. \qquad (27)$$

Numerical evaluation leads to the results plotted in the Figure.



Curve $1 - c'_-$, $2 - c'_+$, $3 - \varkappa'_+$, $4 - \varkappa'_-$

To evaluate the results we need approximate formulas for the case $\mu \ll 1$:

$$(28)$$
$$c' = \sqrt{5/3}\,(1 + 0.1194\mu^2);\quad c'_+ = \sqrt{\mu}\,(1 - 0.3500\mu),$$
$$\varkappa'_- = 0.1807\,\mu^2;\quad \varkappa'_+ = \sqrt{\mu}\,(1 - 0.3500\,\mu).$$

Thus the dispersion equation (25) has two solutions for the velocity and attenuation of small perturbations of the neutral gas.

---

* The sign ± in front of the general square root has been omitted, since the choice of this sign is connected with the choice of the direction of propagation of the wave.

Let us consider the solution c'_-, $\varkappa'_-$ (see the Figure). As $\omega \to 0$, it undergoes transition into expressions known from the theory of continuous media: $c'_- = \sqrt{5/3}$, $\varkappa'_- = 0$. Hence we obtain the usual expression $c = (5RT/3M)^{1/2}$ for the sound velocity in a monatomic gas.

The solution just obtained contains in it the higher approximations of the Chapman-Enskog method (for example, the second approximation of Stokes-Navier, the third approximation of Burnett). In order to be convinced of this, we compare our results with the corresponding theoretical[4-7] and experimental results[7] (see the Table). Thus the solution actually contains in it, as a special case, the various hydrodynamic approximations. The connection assumed in hydrodynamics between the pressure and the velocity gradient is not assumed in our method but is obtained in a special case ($\omega \to 0$).

Comparison of Theoretical values of sound velocity with Experimental
(for argon at 0° and sound velocity $\omega/2\pi =$ 970.68 Kc;
$a = 308.23$ m/sec.

pressure p in cm Hg	Values of c/a (m/sec).			
	experimental	according to Nanier-Stokes	according to Burnett	according to Eq. (27).
78.40	1	1	1	1
47.10	1	1	1	1
22.30	1	1	1	1
13.50	1	1	1	1
9.10	1.002	1	1	1
4.30	1.006	1	1	1.001
3.30	1.008	1.001	1.001	1.002
2.45	1.008	1.001	1.002	1.003
1.80	1.016	1.002	1.002	1.005
1.50	1.028	1.003	1.005	1.007
0.65	1.058	1.019	1.028	1.043
0.34	1.139	1.054	1.103	1.144
0.22	1.259	1.139	1.246	1.281

Increase in the vibration frequency leads to a comparatively large increase in the sound propagation velocity and to a simultaneous increase in the absorption. For sufficiently high frequencies ($\omega \gtrsim 5\omega_0$), $1/\varkappa_- \sim 2\lambda$ (λ = mean free path length). Consequently, this solution gives an analytic description of the Lebedev effect[8] which consists in the presence of a frequency limit for the propagation of ultrasound in neutral gases. This limit is a function of the gas density: the lower the gas density, the lower the frequencies of vibration that cannot be propagated in the gas.

Let us consider the second solution c. The fact of two solutions has already been pointed out in the literature (see, for example, Ref. 3). However, in rarefied gases, only a single wave is observed for small disturbances. This corresponds to the form of the solution c_+', $\varkappa'_+$. Actually, the solution c_+', $\varkappa'_+$ does not give propagating waves in practice: $c'_+ \approx \varkappa'_+$ and the damping $\varkappa'_+$ for $\omega \to 0$ tends to zero as $(\omega/\omega_0)^{1/2}$, while $\varkappa'_-$ tends to zero as $(\omega/\omega_0)^2$.

In conclusion I express my gratitude to Academician N. N. Bogoliubov for his discussion of the results.

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Contribution to the Thermodynamical Theory of the Ferroelectric Properties of Barium Titanate

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The question of the character of the phase transition from the nonferrolectric to the ferroelectric phase is discussed. The influence of an electric field on the temperature for the phase change is considered. It is shown that in the case of a first order phase change, a shift in the transformation point takes place. and the amount of the shift is calculated. The temperature dependence of the dielectric permeability is found for the case of a second order phase change. It is shown that in such a transformation an "induced" piezoeffect" must appear, which consists in the appearance of piezoelectric properties under application of an electric field. The "induced piezomoduli" are calculated.

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1. INTRODUCTION

In spite of the large number of works devoted to the theory of barium titanate, it is still far from completion. This refers, in the first place, to the microtheory of ferroelectric phenomena, but neither can the thermodynamical theory evolved by Ginzburg[1] and developed in Refs. 2-6 be considered fully complete. On the one hand, there are still questions not considered in these works and on the other, there is a need to compare the existing theories among themselves.

In Ref. 1, there are discussed two variants of a theory which is in conformance with the experimentally observed facts. According to the first version, the phase transition at 120° should be considered as a second order phase change, but close to a phase change of first order. According to the second version it should be considered as a first order change, but close to a change of second order. In Refs. 2 and 3, this phase change is considered as a first order change, while in Refs. 4-6, as a second order change.

An attempt was made[7] to settle the question about the character of the phase transformation from the nonferroelectric phase by way of investigation of the dependence of the dielectric permeability near the Curie point on the strength of a steady electric field. The author found that the experimental data are in good agreement with the theory[2] and on that basis concluded that the phase change under investigation must be considered as a change of first order. As will be pointed below, however, (see Sec. 2) experimental data obtained by other authors rather favors the hypothesis that the phase transformation we are speaking of must be regarded as a phase change of second order.

If one again takes into account that even the other experimental results quoted by the various authors differ sharply among themselves (for instance, the thermal behavior of the spontaneous polarization, the constant in the Curie-Weiss law, the jump in specific heat) and depend most strongly on the technique of preparation of the specimens and on their chemical purity, the idea suggests itself that the character of the phase change of which we speak is likewise also determined by these factors and was different in the works of the different authors. The distinction between the theories[1-6] is not only in the question of the character of the phase change under discussion. These theories differ further in that they operate with different thermodynamic functions. In Refs. 1, 4 and 5, the thermodynamic potential is considered, while in Refs. 2 and 6, the free energy is considered.

Just as in the treatment proposed by Landau[8] for phase changes of second order, the conditions of thermodynamic equilibrium of a system are obtained by considering the conditions for a minimum of the appropriate thermodynamic function, written down for a thermodynamically nonequilibrium states. Insofar as in all the works[1-6] the behavior of a crystal is studied in a given electric field and under a given system of stresses (homogeneous), the appropriate thermodynamic function is precisely the thermodynamic potential. Utilization of the free energy, however, and application of the general formulas of the thermodynamics of quasi-statistical processes leads practically to the same results as are obtained by consideration of the thermodynamic optential of nonequilibrium states. As long as the stresses (or deformations) are not taken into account, this is obvious. Writing down nonequilibrium thermodynamic potential in the form $\Phi(P_i, T) = \Psi(P_i, T) - \mathbf{P}\cdot\mathbf{E}$, where $\Psi(P_i, T)$ is a polynomial in powers of P_i, and finding P_i from the condition of a minimum for Φ, we have the equations

$$\partial\Psi/\partial P_i = E_i,\ i = 1, 2, 3,$$

which conform with the thermodynamic identities for quasi-statistical processes, if we take Ψ to be the free energy. In the calculation of the elastic stresses an analogous situation occurs. We satisfy ourselves on this by comparing Refs. 5 and 6.* In Ref. 6 there is used an expansion of some characteristic function Φ into components of the polarization vector P_i and components of the elastic deformation tensor u_{ik}. Although the authors call this function the thermodynamic potential, it is more proper to restrict this designation to the function

$$\zeta = \Phi + \sum_{i,\,k} u_{ik}\,\sigma_{ik},$$

and to regard the function Φ + P · E as the free energy (for quasi-statistical processes), since it complies with the conditions**

$$\partial(\Phi + \mathbf{P}\cdot\mathbf{E})/\partial P_i = E_i; \qquad (1)$$

$$\partial(\Phi + \mathbf{P}\cdot\mathbf{E})/\partial u_{ik} = -\sigma_{ik}.$$

The authors maintain that the resolution into components u_{ik} permits taking account of the effect of electrostriction on the properties of a ferroelectric to a greater degree than the earlier applied expansion 1 of the thermodynamic potential into components σ_{ik}. We cannot agree with such a statement, however, since actually the increases in precision obtained in Ref. 6 are illusory. In order to be convinced of this, it is sufficient to transform the formulas obtained in Ref. 6.

Equations (1) and (2), when developed, have the form:

$$[\alpha + \beta_1 P_x^2 + \beta_2(P_y^2 + P_z^2) + q_{11}u_{xx} + q_{12}(u_{yy} + u_{zz})]\,2P_x + q_{44}(u_{xy}P_y + u_{xz}P_z) = E_x; \qquad (2)$$

$$[\alpha + \beta_1 P_y^2 + \beta_2(P_z^2 + P_x^2) + q_{11}u_{yy} + q_{12}(u_{zz} + u_{xx})]\,2P_y + q_{44}(u_{yz}P_z + u_{yx}P_x) = E_y;$$

$$[\alpha + \beta_1 P_z^2 + \beta_2(P_x^2 + P_y^2) + q_{11}u_{zz} + q_{12}(u_{xx} + u_{yy})]\,2P_z + q_{44}(u_{zx}P_x + u_{zy}P_y) = E_z;$$

$$c_{11}u_{xx} + c_{12}(u_{zz} + u_{yy}) + q_{11}P_x^2 + q_{12}(P_y^2 + P_z^2) = -\sigma_{xx}; \qquad (3)$$

$$c_{11}u_{yy} + c_{12}(u_{xx} + u_{zz}) + q_{11}P_y^2 + q_{12}(P_z^2 + P_x^2) = -\sigma_{yy};$$

$$c_{11}u_{zz} + c_{12}(u_{yy} + u_{xx}) + q_{11}P_z^2 + q_{12}(P_x^2 + P_y^2) = -\sigma_{zz};$$

$$c_{44}u_{xy} + 2q_{44}P_xP_y = -\sigma_{xy};$$

$$c_{44}u_{yz} + 2q_{44}P_yP_z = -\sigma_{yz};$$

$$c_{44}u_{zx} + 2q_{44}P_zP_x = -\sigma_{zx}.$$

Solving the system of Eqs. (3) with respect to u_{ik} and substituting the obtained expressions into Eqs. (2), we obtain a system not essentially different from $(1,4)_5$** in notation and in the signs of the σ_{ik}.

In order that the system Eqs. (2) agree with Eqs. $(1,4)_5$***(accurately with respect to the signs of the σ_{ik}, with regard to which see footnote*) it is necessary to assume

$$\beta_1 = \beta_1' - [q_{11}^2 s_{11} + 4q_{11}q_{12}s_{12} + 2q_{22}^2(s_{11} + s_{12})], \qquad (4)$$

$$\beta_2 = \beta_2' - \beta_1' + (s_{11} - s_{12})(q_{11} - q_{12})^2 - s_{44}q_{44}^2 - 2q_{44}^2/c_{44}.$$

The coefficients denoted by β'_1 and β'_2 in Eqs. (4) are the β_1 and β_2 of Ref. 6. The other symbols have the following meaning:*

$$s_{11} = (c_{11} + c_{12})/(c_{11} + 2c_{12})(c_{11} - c_{12}); \qquad (5)$$

$$s_{12} = c_{12}/(c_{12} - c_{11})(c_{11} + 2c_{12});$$

$$s_{44} = 2/c_{44};$$

---

*We note that the theories[1-6], apart from the difference in the thermodynamic functions utilized, further differ in the extent of the expansions employed and in notation. Thus, in Ref. 2, only one invariant of 6th order is utilized while in Ref. 4 all three of such invariants are taken into account.

**More often, one writes $\partial\Psi/\partial u_{ik} = +\sigma_{ik}$, but owing to the arbitrariness of the condition which establishes the positiveness of the components σ_{ik}, the manner of writing employed in Eq. (1) is also feasible.

**Here and further on, the subscript after an equation number refers to the reference from which the equation is taken.

***Equation (1.4) in Ref. 5 contains misprints.

*The quantities $\varkappa_1$, $\varkappa_2$, $\varkappa_3$ which figure in Ref. 5, appear to correspond, respectively, to the quantities ϑ_{33}, ϑ_{31}, ϑ_{15} from Ref. 6.

$$\varkappa_1 = -(2q_{12}s_{12} + q_{11}s_{11}); \quad (6)$$

$$\varkappa_2 = -[(q_{11} + q_{12})s_{12} + q_{12}s_{11}];$$

$$\varkappa_3 = -2q_{44}/c_{44}.$$

From the agreement of Eqs. (2) with the Eqs. (1,4) of Ref. 5 it follows that their consequences, examined in Refs. 4 and 5 from one side and in Ref. 6 from the other, must likewise also be identical. As a matter of fact, the formulas for the degree of spontaneous polarization, spontaneous deformation, longitudinal and transverse dielectric permeability, piezoelectric moduli and for the dependence of the Curie point on pressure, differ, as can be seen, only in notation**.

Thus, contrary to the opinion of the authors[6], the use of the free energy and its resolution into components u_{ik} is formally equivalent to the use of the thermodynamic potential and its resolution into components σ_{ik}. It would be in error to discern an advantage in the first of these methods in that it allows the determination of the structure of the coefficients β_1 and β_2. In fact β_1 and β_2 must be continuous functions of temperature, while β'_1 and β'_2 can change discontinuously at points of phase transformation, in connection with the discontinuous changes of the elastic moduli s_{ik}. Herein must be found a reflection of the fact that the condition for thermodynamic equilibrium for given temperature and stresses σ_{ik}, is the minimum of the thermodynamic potential, and not that of the free energy.

Completing the comparison of Refs. 4 and 5 with Ref. 6, we note one essential difference in the conclusions. In Ref. 5 it is asserted that for a crystal with $\varkappa_1 > 0$, a unidirectional compression (as well as an elongation) must raise the Curie point, while in Ref. 6 the opposite is maintained. The origin of this disagreement is as such: In Ref. 5 it was shown that the state of a crystal compressed along the z axis with polarization directed also along the z axis does not correspond to the absolute minimum of the thermodynamic potential. The absolute minimum corresponds to the direction of polarization along the x or y axis. Therefore, as ideal crystal must transform under compression into the other phase*, with which the increase in the Curie point is also associated. This effect is not considered in Ref. 6. In a real crystal, the turning of the spontaneous polarization vector is hindered and therefore one can expect "thermal hysteresis". On cooling a previously heated and compressed crystal the transition from the nonferroelectric phase into the ferroelectric must proceed at higher temperatures than the reverse transition in the heating of a crystal compressed along an axis which coincides in direction with the vector P.

2. THE DEPENDENCE OF CERTAIN PROPERTIES OF A CRYSTAL NEAR A PHASE TRANSFORMATION POINT ON THE CHARACTER OF THAT TRANSFORMATION

The question of the properties of a crystal in the neighborhood of the phase transformation point in the presence of an electric field and, in particular, the question of the shift of the transformation point was raised and discussed in Ref. 7. This work, however, does not contain a detailed examination of the question.

In the case of a phase transformation of the first kind there occurs both a shift in the transformation point and a change in the jump of the dielectric permeability at this point. In the case of a phase transformation of the second kind, the phenomenon bears a different character. The fact is, that for a phase transformation of the second kind in the absence of a field, changes in crystal symmetry come about which are also manifested in the appearance of spontaneous polarization. The application of a field, however, promotes the appearance of polarization and consequently, also a change in symmetry at temperatures above the Curie point. That particular Curie point ceases to be more or less discrete in the presence of a field and, strictly speaking it is generally meaningless to speak of a phase transition in this case insofar as all the properties of the crystal vary smoothly with temperature*. In particular, the dielectric permeability also varies continuously, upon which its maximum point is displaced by the applied field. We examine both of these cases, considering terms of sixth degree in the expansion of Φ.

A. PHASE TRANSITION OF THE FIRST KIND

The transformation point is determined from the condition of the equality of the thermodynamic potentials in the nonferroelectric phase I and in the ferroelectric phase II. They have the form**

**The equation for d_{15}, Eq. (20)[6] and the third of Eqs. (4.1)[4] do not agree, apparently due to the inaccuracy of Eq. (20).[6]

*Also tetragonal, but with a different direction of the vector P.

*This circumstance was pointed out to us by V. L. Ginzburg.

**The notation is the same as in Ref. 4.

$$\Phi_{\mathrm{I}} = \Phi_0 + \alpha P_{\mathrm{I}}^2 + {}^1\!/_2\, \beta_1 P_{\mathrm{I}}^4 + {}^1\!/_3\, \gamma_1 P_{\mathrm{I}}^6 - P_{\mathrm{I}} E; \quad \Phi_{\mathrm{II}} = \Phi_0 + \alpha P_{\mathrm{II}}^2 + {}^1\!/_2\, \beta_1 P_{\mathrm{II}}^4 + {}^1\!/_3\, \gamma_1 P_{\mathrm{II}}^6 - P_{\mathrm{II}} E. \tag{7}$$

Setting in the first phase $P_1 = \varkappa E = E/2\alpha$ and restricting ourselves everywhere in the calculations to terms which contain E no higher than to the second power, we get

$$\Phi_{\mathrm{I}} = \Phi_0 - E^2 / 4\alpha. \tag{8}$$

Comparing Eqs. (7) and (8) and taking into consideration that P_{II} satisfies the equation

$$(\gamma_1 P_{\mathrm{II}}^4 + \beta_1 P_{\mathrm{II}}^2 + \alpha)\, 2P_{\mathrm{II}} = E, \tag{9}$$

we obtain

$${}^1\!/_6\, \beta_1 P^4 + {}^2\!/_3\, \alpha P^2 - {}^5\!/_6\, PE = -E^2 / 4\alpha \tag{10}$$

(here and below, the index II of P is dropped).

Assuming $P \gg E$, the solution of Eq. (10) can be written in the form

$$P = \sqrt{-4\alpha / \beta_1} + CE + DE^2, \tag{11}$$

$$C = -{}^5\!/_8\, \alpha, \qquad D = -(51 / 128\, \alpha^2) \sqrt{-\beta_1 / 4\alpha}. \tag{12}$$

On the other hand, the solution of Eq. (9) can, under the same assumption, be written in the form

$$P = P_0 + AE + BE^2, \tag{13}$$

$$A = -{}^1\!/_4\, (\beta_1 P_0^2 + 2\alpha)^{-1}. \qquad B = -A^2 (7\beta_1^2 P_0^2 + 10\alpha) / P_0 (2\beta_1 P_0^2 + 4\alpha). \tag{14}$$

Here P_0 is the solution to Eq. (9) for $E = 0$:

$$P_0^2 = (-\beta_1 + \sqrt{\beta_1^2 - 4\alpha\gamma_1}) / 2\gamma_1. \tag{15}$$

At the transformation point, the values of P determined from Eqs. (11) and (13) must coincide. In the calculation of the shift of the transformation point, we restrict ourselves to first order in E. Then for the determination of the amount of this shift we obtain

$$\sqrt{-4\alpha / \beta_1} + CE = P_0 + AE. \tag{16}$$

We denote α and β_1 at the transformation point for $E = 0$ by α_0 and β_{10}, and the value of P_0 at this point by P_{00}:

$$P_{00} = \sqrt{-4\alpha_0 / \beta_{10}} = \sqrt{-3\beta_{10} / 4\gamma_1}. \tag{17}$$

Denoting the transformation point by T_c, and the shift of the transformation point by ΔT_c, expanding α and β_1 into series in ΔT_c near α_0 and β_{10} and restricting ourselves to linear terms, we find after substitution of these expansions in Eq. (16)

$$\Delta T_c = E / P_{00} (\dot{\alpha} - 2\alpha_0 \dot{\beta}_1 / \beta_{10}), \tag{18}$$

where α and β_1 are the values of the derivatives with respect to T at the point $(T_c)_{E=0}$. Thus the sign of the shift of the transformation point is determined by the sign of the quantity $(\alpha - 2\alpha_0 \beta_1/\beta_{10})$. For $BaTiO_3$, as numerical estimates show (see below), the second term is considerably smaller in absolute value than the first. Consequently the transformation point is shifted in the direction of higher temperatures by the field and the magnitude of the shift is equal to

$$\Delta T_c \approx E / P_{00} \dot{\alpha}. \tag{19}$$

We now find the jump of $\varkappa$ at the transformation point. First of all, from Eq. (13) we find

$$\varkappa = (\varepsilon - 1) / 4\pi = (\partial P / \partial E)_T = A + 2BE. \tag{20}$$

In order to find the value of $\varkappa$ for $T = T_c$ on the low temperature side it is necessary to expand A and B into series in ΔT and to substitute the value of ΔT_c from Eq. (18), whereupon it is obviously sufficient for the calculation of B to terminate with terms of zero degree, and for the calculation of A, with terms linear in ΔT_c. Performing the expansion, we find

$$A = \frac{1}{8\alpha_0}\left[1 + \Delta T_c \left(2\,\frac{\dot{\alpha}}{\alpha_0} - \frac{6\dot{\beta}_1}{\beta_{10}}\right)\right]; \qquad B = -\frac{9}{128\,\alpha_0^2}\sqrt{-\frac{\beta_{10}}{4\alpha_0}}. \tag{21}$$

Substituting the value of ΔT_c in A, we find from Eq. (20)

$$\varkappa_{T_c} = \frac{1}{8\alpha_0} + \frac{E}{4\alpha_0^2}\sqrt{\frac{-\beta_{10}}{4\alpha_0}}\left[\frac{5}{16} - \frac{\dot{\beta}_1 / \beta_{10}}{\alpha / \alpha_0 - 2\beta_1 / \beta_{10}}\right]. \tag{22}$$

We find the value of $\varkappa$ on the high temperature side from

$$(\varkappa)_{T_c} = \left(\frac{1}{2\alpha}\right)_{T_c} = \frac{1}{2\alpha_0}\left(1 - \frac{\alpha}{\alpha}\,\Delta T_c\right).$$

Substituting here ΔT_c from Eq. (18); we get

$$\varkappa_{T_c} = \frac{1}{2\alpha_0} - \frac{\dot{\alpha}}{2\alpha_0^3}\sqrt{-\frac{\beta_{10}}{4\alpha_0}}\,\frac{E}{\dot{\alpha}/\alpha_0 - 2\dot{\beta}_1/\beta_{10}}. \qquad (23)$$

From Eqs. (22) and (23) it is seen that application of an electric field somewhat reduces the jump of $\varkappa$ at the transformation point, which conforms in the absence of a field to the "rule of four"[1].

B. PHASE TRANSITION OF THE SECOND KIND

As was explained above, in the application of a field, the phase transition is "washed out". In this case it is meaningless to speak of the dielectric permeability at the transition point, but it is expedient to set up the problem of finding P and ϵ for temperatures in the neighborhood of the Curie point*[10]. Since in this temperature range one can disregard the term in P^5 in Eq. (9), the problem reduces to the finding of the solution of the equation

$$\beta_1 P^3 + \alpha P = E/2 \qquad (24)$$

and to finding the susceptibility

$$\varkappa = (\partial P/\partial E)_T = {}^1/_2(3\beta_1 P^2 + \alpha)^{-1}. \qquad (25)$$

The solution of Eq. (24) can be easily studied by means of tables.[9] The dependence of $\varkappa$ on E for T=const. is seen immediately from Eq. (25):

$$(\partial\varkappa/\partial E)_T = -3\beta_1 P/2(3\beta_1 P^2 + \alpha)^3 < 0. \qquad (26)^{**}$$

For a fixed temperature $\varkappa$ decreases with an increase in E.*** The detailed dependence $\varkappa(E)$ can be followed by the use of the tables, but we are confining ourselves only to the observations made.

The function $\varkappa(T)$ for E=const. is of great interest.

The function $y(q)$, is shown in Fig. 1 where

$$y = \varkappa(\beta_1 E^2/4)^{1/3}; \qquad 4\alpha^3/\beta_1 E^2 = q,$$

are plotted by use of the tables.[9] From the graph it is seen that y has a maximum ≈0.21 for q≈1.2.

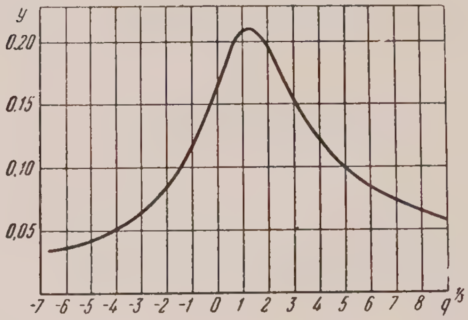

FIG. 1

We disregard the temperature dependence of β_1. Then the relation obtained can be written in the following form:

$$\varkappa_{\max}(\beta_1 E^2/4)^{1/3} \approx 0{,}33; \qquad (27)$$

$$(\alpha)_{\varkappa=\varkappa_{\max}} \approx 0\ 75\,(\beta, E^2)^{1/3}.$$

Thus, application of a field displaces the point of maximum $\varkappa$ into the region of higher temperatures, whereupon this displacement is proportional to $E^{2/3}$, while the maximum value of $\varkappa$ is inversely proportional to $E^{2/3}$. If we denote the amount of the shift of temperature which corresponds to the maximum, by $\Delta T = T_{\max} - T_c$ and assume for α a linear dependence on temperature, $\alpha = \dot{\alpha}(T - T_c)$, we can get from Eq. (27)

$$\varkappa_{\max}\,\Delta T = \text{const} \approx 0.25/\dot{\alpha}. \qquad (28)$$

With β_1 temperature dependent, these simple relations no longer hold. In this case however, it is not difficult to calculate the function $\varkappa(T)$ for E=constant by means of the tables. In Fig. 2 several curves for various values of E are shown. In their construction, a linear dependence of α and β_1 on temperature was assumed*:

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*By the Curie point, naturally, is understood the temperature which corresponds to the phase transition for E=0, i.e., the temperature for which α=0.

** The positiveness of the denominator in Eq. (26) follows from the fact that the solution of Eq. (24) satisfies minimum Φ.

*** The decrease of $\varkappa$ with increase in E has been noted in the literature.[1]

*The value of the constant in α is known from Ref. 10. As regards the coefficient in β_1, using the numbers given, a satisfactory agreement of several theoretical formulas[4] with the experimental results[11] was successfully obtained.

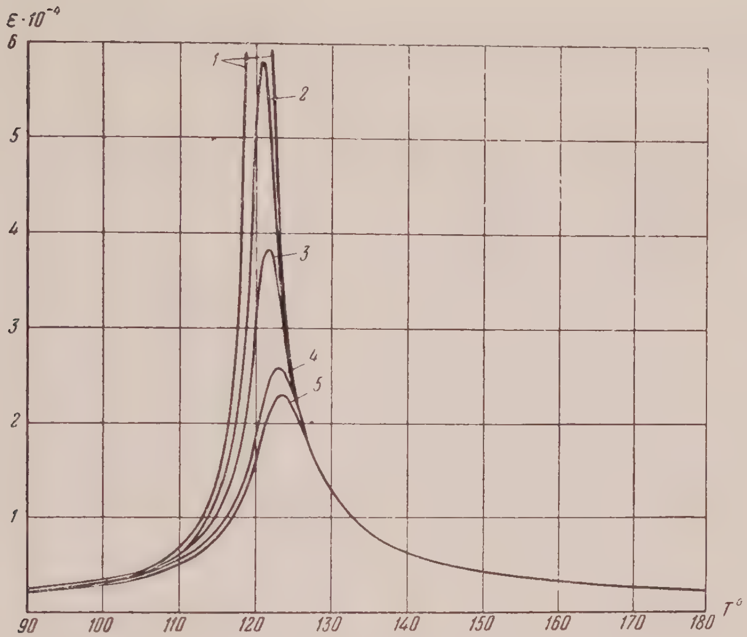

FIG. 2 1 — $E = 0$, 2 — $E = 500$, 3 — $E = 1000$, 4 — $E = 2000$, 5 — $E = 2500$ V/см

$$\alpha = 5\cdot 10^{-5}(t - 120);$$

$$\beta_1 = [1.4\cdot 10^{-6} - 1.9\cdot 10^{-7}(t - 120)]\ \mathrm{cm}^4/\mu\ \mathrm{sec}^2.$$

The curves in Fig. 2, in their qualitative shape, are very close to the experimental curves obtained in Ref. 12. In any event, it appears very probable that these latter curves can be interpreted only regarding the phase change as a transition of the second kind. Unfortunately, a quantitative comparison is not possible since the curves of Ref. 12 were obtained for polycrystals and also since this work does not contain a sufficiently complete description of the properties of the specimens studied.

3. INDUCED PIEZOEFFECT

In the neighborhood of the Curie point, a crystal in which a phase transformation of the second kind occurs must be "friable", i.e., very compliant with respect to external influences. One can expect that one of the manifestations of this friability should be an effect where a crystal which, at a given temperature, does not have piezoelectric properties, acquires them upon application of an electric field. We shall call such an effect an induced piezoeffect. We maintain the same designation also for the more general case when a crystal, before the application of an electric field, possesses a piezoeffect characterized by a certain matrix of piezomoduli and application of a field leads to the emergence of new piezomoduli (and to the modification of the old ones).

The phenomenon of induced piezoeffect should also be observed for a $BaTiO_3$ crystal near the Curie point, both above and below this point.

Let the stress component σ_{kl}, in the presence of the electric field component E_j, give rise to the appearance of the induced polarization

$$(P_i)_{\mathrm{ind}} = C_{ijkl}\,\sigma_{kl}\,E_j.$$

We shall call the coefficients C_{ijkl} (symmetrical in the indices k,l) the induced piezomoduli. They are easy to find by use of the formulas of Ref. 5, where the components of the polarizability tensor $\varkappa_{ij}$ in the presence of elastic stresses σ_{kl} are calculated. In particular

$$C_{ijkl} = \partial^2 P_i / \partial E_j \partial\sigma_{kl} = \partial\varkappa_{ij} / \partial\sigma_{kl}.$$

In actuality, however, the calculation of the induced piezomoduli according to the formulas of Ref. 5 still does not give correct expressions for the induced piezomoduli, since in the expansion of the thermodynamic potential used in Ref. 5, certain terms which play an essential role in the investigation of the question were not considered. On taking them into consideration the thermodynamic potential of a crystal with cubic symmetry assumes the form:

$$\Phi = \Phi_0 + (\alpha_1 P_x^2 + \alpha_2 P_y^2 + \alpha_3 P_z^2) \tag{29}$$

$$+ {}^1\!/_2\, \beta_1 P^4 + \beta_2 (P_x^2 P_y^2 + P_y^2 P_z^2 + P_z^2 P_x^2)$$

$$- (P_x E_{x1} + P_y E_{y1} + E_{z1} P_z)$$

$$- 2\varkappa_3 (\sigma_{12} P_x P_y + \sigma_{23} P_y P_z + \sigma_{31} P_z P_x)$$

$$- 2\delta_3 [\sigma_{12} (P_x E_y + P_y E_x)$$

$$+ \sigma_{23} (P_y E_z + P_z E_y) + \sigma_{31} (P_z E_x + P_x E_z)]$$

$$- {}^1\!/_2\, s_{11} (\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2)$$

$$- s_{12} (\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11})$$

$$- {}^1\!/_2\, s_{44} (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2),$$

where the notation

$$E_{x1} = E_x + [\delta_1 \sigma_{11} + \delta_2 (\sigma_{22} + \sigma_{33})] E_x; \tag{30}$$

$$E_{y1} = E_y + [\delta_1 \sigma_{22} + \delta_2 (\sigma_{33} + \sigma_{11})] E_y;$$

$$E_{z1} = E_z + [\delta_1 \sigma_{33} + \delta_2 (\sigma_{11} + \sigma_{22})] E_z.$$

is used. Here δ_1, δ_2 and δ_3 are the coefficients with the invariants not treated in Ref. 5. The remaining notation agrees with that used in Ref. 5.

The necessary conditions for a minimum of Φ have the form:

$$2P_x [\alpha_1 + \beta_1 P^2 + \beta_2 (P_y^2 + P_z^2) \tag{31}$$

$$= E_{x1} + 2\varkappa_3 (\sigma_{12} P_y + \sigma_{31} P_z)$$

$$+ 2\delta_3 (\sigma_{12} E_y + \sigma_{31} E_z)$$

etc. In order to determine the induced piezomoduli from this, it is convenient to calculate them one after another. Thus, for the calculation of C_{1111} we set $\sigma_{11} \neq 0$, and all the remaining components σ_{kl} equal to zero and $E_x \neq 0$, $E_y = E_z = 0$. We examine, at first, temperatures above and not too close to the Curie point. Then it is possible to assume

$$P_x = \varkappa_{11} E_x + C_{1111} E_x \sigma_{11}; \qquad P_y = P_z = 0.$$

Substituting this in Eq. (31) and taking into consideration that $\varkappa_{11} = 1/2\,\alpha$, we find, disregarding terms of higher order

$$C_{1111} = \delta_1 / 2\alpha + \varkappa_1 / 2\alpha^2. \tag{32}$$

The remaining piezomoduli are calculated in an analogous way.

The results obtained can be formulated in the form of the following rules.

1. Only those induced piezomoduli are different from zero for which all indices consist of repeated numbers. Piezomoduli, the indices of which contain some number only once, are equal to zero.
2. The nonzero induced piezomoduli are expressed by a two term formula, the first term of which is proportional to $1/\alpha$, and the second proportional to $1/\alpha^2$
3. The coefficients with $1/\alpha$ and $1/\alpha^2$ for the moduli corresponding to elongations (compressions), are equal to $\delta_1 / 2$ and $\varkappa_1 / 2$ respectively, when the elongation (compression) coincides with the direction of the electric field, and are equal to $\delta_2 /2$ and $\varkappa_2 /2$, respectively when the elongation (compression) is perpendicular to the direction of the electric field.
4. The coefficients with $1/\alpha$ and $1/\alpha^2$ for the moduli corresponding to shear are equal to $\delta_3 / 2$ and $\varkappa_3 /4$, respectively.

We now consider temperatures below the Curie point. The calculation of the induced piezomoduli is carried out analogously to the case where $T > T_c$. The difference is merely in that the general piezomoduli d_{ikl} are now not equal to zero and a spontaneous polarization P_0 exists. With nonzero components E_j and σ_{kl} and the choice of the direction of spontaneous polarization along the z axis, there should be substituted in Eq. (31):

$$P_x = \varkappa_{1j} E_j + d_{1kl} \sigma_{kl} + C_{1jkl} E_j \sigma_{kl} \tag{33}$$

and analogous expressions for P_y and P_z. The results of the calculation reduce to the following: 1) as in the case for $T > T_c$, all of the induced piezomoduli whose indices contain even a single nonrepeated number are equal to zero. 2) Replacement of the index 1 by 2, and vice versa, in the first and second pair of indices does not change the value of the modulus. The essentially different, nonzero, induced piezomoduli are equal to

$$\begin{gathered}
C_{3333} = -\delta_1 / 4\alpha - \varkappa_1 / 4\alpha^2; \\
C_{3311} = -\delta_2 / 4\alpha - \varkappa_2 / 4\alpha^2; \\
C_{1133} = -\delta_2\beta_1 / 2\beta_2\alpha \\
+ (\beta_1^2 / 2\beta_2^2\alpha^2)\,[\varkappa_2 - \varkappa_1 (\beta_1 + \beta_2) / \beta_1]; \\
2C_{1313} = -\delta_3\beta_1 / \beta_2\alpha - \varkappa_3 (2\beta_1 + \beta_2)\,\beta_1 / 4\beta_2^2\alpha^2; \\
C_{1111} = -\delta_1\beta_1 / 2\beta_2\alpha \\
+ (\beta_1 / 2\beta_2^2\alpha^2)\,[\varkappa_1\beta_1 - \varkappa_2 (\beta_1 + \beta_2)]; \\
C_{1122} = -(\beta_1 / 2\alpha\beta_2)\,(\delta_2 + \varkappa_2 / \alpha); \\
2C_{2121} = -\delta_3\beta_1 / \beta_2\alpha + \varkappa_3\beta_1^2 / 2\beta_2^2\alpha^2; \\
2C_{3131} = -\delta_3 / 2\alpha - \varkappa_3 (\beta_2 + 2\beta_1)\,\beta_1 / 4\alpha^2\beta_2^2.
\end{gathered} \tag{34}$$

In conclusion we note that there must also exist an effect inverse to the induced piezoeffect. From the thermodynamic formula*

$$d\Phi = -P_i\,dE_i - u_{kl}\,d\sigma_{kl} \tag{35}$$

there follows the well-known relation

$$\partial P_i / \partial\sigma_{kl} = \partial u_{kl} / \partial E_i. \tag{36}$$

It contains in itself, along with the inverse piezoeffect also the inverse induced piezoeffect. Substituting in Eq. (36)

$$(P_i)_{\text{ind}} = \varkappa_{ij}E_j + d_{ikl}\sigma_{kl} + C_{ijkl}E_j\sigma_{kl}, \tag{37}$$

we find

$$u_{kl} = (u_{kl})_0 + d_{ikl}E_i + C_{ijkl}E_iE_j. \tag{38}$$

Thus the effect inverse to the induced piezoeffect is electrostriction and the coefficients C_{ijkl} are the coefficients of electrostriction.[6]

For deformations of elongation (compression) there correspond $k=l$; but then, also $i=j$ (otherwise $C_{ijkl} = 0$). Consequently, such deformations depend quadratically on the component of field intensity. To shear deformations there correspond $k \neq l$; but then, $i=k$ and $j=l$. Consequently, shear deformations arise only in the presence of two components. In the general case when the directions of the vectors P_0 and E make an arbitrary angle, both extension (compression) and shear occur.

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*Here, we keep in mind the equilibrium thermodynamic potential.

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Statistics of Electrons and Holes in a Homopolar Semiconductor Taking into Account their Interaction with the Lattice Vibrations

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The interaction between electrons and phonons in a homopolar semiconductor is taken into account in the examination of the stationary electron distribution, and is not considered as being merely the cause of transitions between stationary states of "free" electrons and phonons. The limiting cases of high and of low temperatures are investigated. It is shown that because of the interaction between electrons and phonons a perfectly pure semiconductor has a conductivity different from zero at the temperature of absolute zero.

IN the existing theory of homopolar semiconductors the interaction of the current carriers with the vibrations of the crystalline lattice is taken into account only in the consideration of nonequilibrium processes: it is considered that it only induces transitions between the stationary states of electrons in a rigid lattice, determining for example Joule heat losses, etc. However, such a procedure is not entirely logical, since actually the interaction with the lattice vibrations always takes place; consequently, in taking it into account as the cause which induces the transitions we—in the same approximation—destroy the consistency of the concept of the states between which these transitions take place. The above means that strictly speaking the interaction between electrons and phonons should be taken into account even in the determination of the energy spectrum and of the density matrix for the electrons in the lattice, i.e., in the solution of the equilibrium problem. It should be emphasized that the necessity to do so arises logically not only in the case of strong coupling between the electrons and the phonons,[1,2] but also in the case of weak coupling which may be taken into account by means of some form of perturbation theory. As will be seen below, in this last case the interaction of electrons and holes with the lattice vibrations leads to changes in the density matrix which are not large quantitatively, but which are (at low temperatures) of considerable significance in principle.

Unfortunately the formulation of the problem in a general form suitable for arbitrary temperatures is very complicated. Therefore we shall restrict ourselves here only to the limiting cases of high and of low temperatures (strictly speaking in the second case we will be concerned with a semiconductor at absolute zero). We shall consider a semiconductor having intrinsic conductivity and with spherical surfaces of constant energy; we will take into account only the interaction with the acoustic lattice vibrations and, for the sake of simplicity, we shall take the coupling constant to be the same both for the conduction electrons and for the holes. Naturally such a model is not too suitable for exact calculations (in particular the zero point optical vibrations should, without any doubt, be taken into account); however, the principle of the fundamental result which we shall obtain will not be affected by these approximations.

Under the above approximations there exists no interaction with transverse waves and the Lagrangian for the interaction between the electrons and the longitudinal acoustic lattice vibrations has the form[3] (we use the interaction representation and we set $h = c = 1$, where c is the velocity of sound):

$$L_{\text{int}} = \int L(x)\,dx \tag{1}$$

$$L(x) = \Big\{ g \sum_s \bar{\psi}_s(x)\,\psi_s(x) + \rho(x) \Big\}\,\varphi; \tag{2}$$

$$\varphi = \partial A / \partial x_0.$$

Here $\psi(x)$ and $A(x)$ are the quantized wave functions of the "noninteracting" electron and phonon fields, $\rho(x)$ is the "external charge density", $x = \{\mathbf{x}, x_0\}$, x_0 is the time, s is the spin index, g is the coupling constant (which in the system of units adopted has the dimensions of length). In the usual units g has the form:

$$g = {}^2/_3\,\pi^{3/2} C c^{-2} \sqrt{V_0 / M}, \tag{3}$$

where V is the volume of the elementary cell, M is the mass of the atom of the lattice, C is a constant which occurs in the theory of electrical conductivity.[4]

Inasmuch as all the operators in which we will be interested will turn out to be diagonal in the spin indices the latter will not be explicitly indicated in what follows.

2. SEMICONDUCTOR AT ABSOLUTE ZERO

The "single particle" density matrix for the electrons and holes at absolute zero can be found most simply by calculating Green's function for the electron field interacting with phonons. The appropriate methods were developed in an earlier paper of the author.[5] As was shown there,[5] the Green's function for the electrons

$$G(x, y) = \frac{i}{\langle S\rangle_0}\langle T\{\psi(x)\overline{\psi}(y)S\}\rangle_0$$

is determined by the equation

$$G(x, y) = iK(x, y) - ig\int dz K(x, z)\, G(z, y)\, a(z) - i\int dz\, dx' K(x, z)\, \Delta E(z, x')\, G(x', y), \tag{4}$$

where

$$\Delta E(x, z') = ig\int dz'\, dz'' \frac{\delta a(z')}{\delta\rho(z)} G(z, z'') \frac{\delta G^{-1}(z'', z')}{\delta a(z')}, \tag{5}$$

$$a(x) = -i\delta \ln\langle S\rangle_0 / \delta\rho(x), \tag{6}$$

in which S is the S-matrix, T is the symbol representing a T-product, $\langle\ldots\rangle_0$ is an average over the "ground state in the absence of interaction."

The expansion of ΔE in a series in powers of the coupling constant leads in the second order in g to the following linear equation for the Green's function [for $\rho(x) = 0$]:

$$G(x, y) = iK(x, y) - g^2\int dz\, dz' K(x, z)\, F(z', z)\, K(z', z)\, G(z', y), \tag{7}$$

$$K(x, y) = \langle T\{\psi(x)\overline{\psi}(y)\}\rangle_0,$$

$$F(x, y) = \langle T\{\varphi(x)\varphi(y)\}\rangle_0.$$

In the present case we have

$$F(x, y) = \frac{1}{2\pi^3}\int d\mathbf{f} f \exp\{i(\mathbf{f}, \mathbf{x} - \mathbf{y}) - if\,|x_0 - y_0|\} \tag{8}$$

($\mathbf{f}$ is the quasimomentum of the phonon, $f \leq f_0$, where f_0 is the Debye upper limit);

$$\psi(x) = (2\pi)^{-3/2} \times\int d\mathbf{p}\,\{\exp\{i\mathbf{p}\mathbf{x} - iW_c(\mathbf{p})x_0\}\, a(\mathbf{p}) + b^*(\mathbf{p})\exp\{i\mathbf{p}\mathbf{x} - iW_v(\mathbf{p})x_0\}\}, \tag{9}$$

a^*, a, b^*, b are the Fermi operators for the creation and annihilation of conduction electrons and of holes in the valence band;

$$W_c(\mathbf{p}) = W_0 + \mathbf{p}^2/2m^*, \qquad W_v(\mathbf{p}) = -\mathbf{p}^2/2m^{**}, \tag{10}$$

W_0 is the width of the forbidden band (calculated in the absence of interaction with phonons), m^* and m^{**} are respectively the effective masses of the conduction electron and of the hole in the valence band (in a rigid lattice).

Nocing that

$$\langle a^*(p)\, a(p)\rangle_0 = \langle b^*(p)\, b(p)\rangle_0 = 0, \tag{11}$$

we obtain

$$K(x, y) = \lim_{\varepsilon\to 0}\frac{i}{(2\pi)^4}\int d\mathbf{p} e^{i(\mathbf{p}, \mathbf{x}-\mathbf{y})} \times\int_{-\infty}^{+\infty} dp_0 e^{-ip_0(x_0-y_0)} K_\varepsilon(\mathbf{p}, p_0); \tag{12}$$

$$K_\varepsilon(\mathbf{p}, p_0) = (2p_0 - W_c - W_v)/(p_0 - W_c + i\varepsilon) \times (p_0 - W_v - i\varepsilon).$$

Equation (7) can be solved easily. Setting

$$G(x, y) = \lim_{\varepsilon\to 0}\frac{i}{(2\pi)^4}\int d\mathbf{p}\, dp_0 \times \exp\{i(\mathbf{p}, \mathbf{x} - \mathbf{y}) - ip_0(x_0 - y_0)\}\, G_\varepsilon(\mathbf{p}, p_0), \tag{13}$$

we obtain

$$G_\varepsilon(\mathbf{p}, p_0) = \frac{i(2p_0 - W_c - W_v)}{(p_0 - W_c + i\varepsilon)(p_0 - W_v - i\varepsilon) + ig^2(2p_0 - W_c - W_v) f(\mathbf{p}, p_0)}, \tag{14}$$

$$\begin{aligned} f(\mathbf{p}, p_0) &= \int F(-x) K(x) e^{-i\mathbf{p}\mathbf{x} + ip_0x_0} d\mathbf{x}\, dx_0 \qquad (15) \\ &= \frac{im^*}{\pi^2 p}\Big\{-\frac{16}{3}(1+\gamma^2) m^* f_0 p - \frac{2}{3} f_0^2 p (1+\gamma) \\ &+ \frac{f_0^3}{3} \ln\left|\frac{(f_0 - p - m^*)^2 - a^2}{(f_0 + p - m^*)^2 - b^2}\right| + \gamma \frac{f_0^3}{3} \ln\left|\frac{(f_0 - p - m^{**}) - c^2}{(f_0 + p - m^{**}) - d^2}\right| \\ &+ a[a^2 + (p+m^*)^2] \ln\left|\frac{f_0 - p - m^* + a}{f_0 - p - m^* - a}\,\frac{p + m^* + a}{p + m^* - a}\right| \\ &+ \gamma c[c^2 + (p+m^{**})^2] \ln\left|\frac{f_0 - p - m^{**} + c}{f_0 - p - m^{**} - c}\,\frac{p + m^{**} + c}{p + m^{**} - c}\right| \\ &+ \left[\frac{(p+m^*)^3}{3} - a^2(p+m^*)\right] \ln\left|\frac{(f_0 - p - m^*)^2 - a^2}{(p+m^*)^2 - a^2}\right| \\ &+ \gamma\left[\frac{(p+m^{**})^3}{3} - c^2(p+m^{**})\right] \ln\left|\frac{(f_0 - p - m^{**})^2 - c^2}{(p+m^{**})^2 - c^2}\right| \\ &- \left[\frac{(-p+m^*)^3}{3} - b^2(-p+m^*)\right] \ln\left|\frac{(f_0 + p - m^*)^2 - b^2}{(-p+m^*) - b^2}\right| \\ &- \gamma\left[\frac{(-p+m^{**})^3}{3} - c^2(-p+m^{**})\right] \ln\left|\frac{(f_0 + p - m^{**})^2 - d^2}{(-p+m^{**})^2 - d^2}\right| \\ &- b[b^2 + (-p+m^*)^2] \ln\left|\frac{f_0 + p - m^* + b}{f_0 + p - m^* - b}\,\frac{p - m^* - b}{p - m^* + b}\right| \\ &- \gamma d[d^2 + (-p+m^{**})^2] \ln\left|\frac{f_0 + p - m^{**} + d}{f_0 + p - m^{**} - d}\,\frac{p - m^{**} - d}{p - m^{**} + d}\right|\Big\}. \end{aligned}$$

To reduce the length of the above expression, the following notation has been introduced

$$\begin{aligned} a^2 &= 2m^*\left(p_0 + p + \frac{m^*}{2} - W_0 + i\varepsilon\right); \qquad (16) \\ b^2 &= 2m^*\left(p_0 - p + \frac{m^*}{2} - W_0 + i\varepsilon\right); \\ c^2 &= 2m^{**}\left(p_0 + p + \frac{m^{**}}{2} - i\varepsilon\right); \\ d^2 &= 2m^{**}\left(p_0 - p + \frac{m^{**}}{2} - i\varepsilon\right); \\ \gamma &= m^{**}/m^*; \quad p = |\mathbf{p}|. \end{aligned}$$

The "single electron" and the "single hole" density matrices are obtained from (13) by means of formula (2.5) of the author's earlier paper.[5]

We note that in order of magnitude we have (if m^*, m^{**} are of the order of an electron mass, and the width of the forbidden band is approximately 1 ev): $f_0 \sim 10^8\ \mathrm{cm}^{-1}$; $m^*, m^{**} \sim 10^5\ \mathrm{cm}^{-1}$; $W_0 \sim 10^{10}\ \mathrm{cm}^{-1}$. Therefore, approximately,[++]

$$f(\mathbf{p}, p_0) \approx -2im^* f_0^2 (1+\gamma)/\pi^2. \tag{17}$$

The integral over p_0 in (13) can easily be evaluated, and we obtain for the "single electron" density matrix in the p-representation the expression

$$\begin{aligned} &\rho(\mathbf{p}, \mathbf{p}') \qquad (18) \\ &= \delta(\mathbf{p} - \mathbf{p}')\, 2\beta^2 \left\{\left[p^2 \frac{m^* + m^{**}}{2m^* m^{**}} + W_0\right]^2 + 4\beta^4\right\}^{-1/2}; \end{aligned}$$

$$\beta^2 = 2\pi^{-2} g^2 m^* f_0^2 (\gamma + 1). \tag{19}$$

The poles of the function (14), W_1 and W_2, determine (for $\epsilon \to 0$) the spectrum of the elementary excitations of the system. We have

$$\begin{aligned} W_1 &= {}^1/_2\{W_c + W_v - 2\beta^2 \qquad (20) \\ &\quad + \sqrt{(W_c - W_v)^2 + 4\beta^4}\}, \\ W_2 &= {}^1/_2\{W_c + W_v - 2\beta^2 \\ &\quad - \sqrt{(W_c - W_v)^2 + 4\beta^4}\}, \end{aligned}$$

[++] Since this expression diverges as $f_0 \to \infty$, it is clear that high frequency phonons play the principal role, and that therefore all our results have only a highly qualitative meaning.

As in the unperturbed problem we obtain here two allowed bands separated by a forbidden region of width

$$\Delta W = \sqrt{W_0^2 + 4\beta^4}. \tag{21}$$

Thus, as should be expected, the excitation spectrum has a typical "semiconductor" form. This to a certain extent justifies the qualitative conclusions of the band theory (as applied to the present case). However, it is essential to note that, as may be seen from (18), due to the interaction between electrons and phonons the density matrix does not have a "stepwise" character. As is well known,[6] this suffices to produce the result that an arbitrarily small external electric field will give rise to a current. We emphasize that we are now discussing the ground state, and that consequently by definition the current carriers in the present case are not the elementary excitations which might arise by thermal means in an equilibrium system. (The latter naturally might also have served as current carriers, but at absolute zero they are absent.)

One may without difficulty compute the "effective concentration" n^* of the current carriers in the ground state of the semiconductor under consideration. In our approximation

$$n^* \approx (2/3\pi^2)\, p_m^3 \,(1 + W_0^2/4\beta^4)^{-1/2}, \tag{22}$$

where p_m is a certain maximum value of the absolute magnitude of the quasimomentum of the electron.

A precise evaluation of n^* is difficult, as it is very difficult to determine p_m. It is only clear that under the ordinary experimental conditions the quantity n^* is small compared to the density of the excitation of the semiconductor type and does not manifest itself in any noticeable way. However, at sufficiently low temperatures, when the conductivity determined by current carrying excitations (and which apparently is the only type that has been observed until now) practically disappears, the resistance of the semiconductor should not increase without limit: because of the presence of current carriers in the ground state the conductivity of the semiconductor must remain finite (though small). In other words, because of the interaction between the electrons and the zero point vibrations of the lattice the semiconductor becomes "metallized" at low temperatures. We emphasize that this refers to any arbitrary semiconductor, and is not connected with the presence of impurities. Unfortunately we do not know of any experimental work the results of which might be unambiguously interpreted along the lines of the arguments presented above. In our opinion the carrying out of appropriate investigations might be of definite interest, although apparently this would not be a simple matter (single crystals should be used which do not have any impurities in order to avoid effects connected with "impurity bands"[7]).

3. HOMOPOLAR SEMICONDUCTOR AT HIGH TEMPERATURES

At high temperatures, taking into account the interaction between the electrons and the phonons in a consistent way does not lead to such fundamentally significant results as at low temperatures: however, it is apparently necessary to take it into account for the investigation of kinetic processes in semiconductors without the use of the kinetic equation. Indeed, it is well known (see, for example Refs. 8,9) that the criterion for the applicability of the kinetic equation is fulfilled rather poorly in semiconductors. If we desire to construct a theory without making use of the kinetic equation it seems to us that we should first of all investigate the equilibrium properties of the system of conduction electrons (and holes) interacting with phonons. This section is devoted to the above topic.

Following Landau and Lifshitz[10] we shall say that those temperatures are high for which kT appreciably exceeds the average energy of the interaction between electrons and phonons (evaluated per electron). In this case in order to investigate equilibrium systems of electrons interacting with lattice vibrations it is natural to use thermodynamic perturbation theory[10] with the one difference that in this case it turns out to be more convenient to compute not the free energy F, but the thermodynamic potential $\Omega = F - \mu N$, where μ is the chemical potential and N is the average number of particles in the system. The expression for Ω (including terms up to second order) is obtained in the same way as the corresponding formula for F (see Ref. 10, Sec. 32):

$$\Omega = \Omega_0 + \frac{1}{2}\sum_{n,\,m}\frac{(n\,|\,H_{\text{int}}\,|\,m)^2}{E_n^{(0)} - E_m^{(0)}}\left\{\exp\frac{\Omega_0 - E_n^{(0)}}{kT} - \exp\frac{\Omega_0 - E_m^{(0)}}{kT}\right\} + \frac{\overline{H}_{\text{int}}^2}{2kT}. \tag{23}$$

Here Ω_0 is the thermodynamic potential of the system of "free" electrons and phonons; n, m are the sets of indices (in this case the occupation numbers of the electron and phonon states) which characterize the states of the unperturbed system with energies $E_n^{(0)}$ and $E_m^{(0)}$. The Hamiltonian of the interaction H_{int} is obtained from (1) and (2), it being sufficient in the present case to consider separately only the conduction electrons or the holes.

Setting

$$\psi(x) = (2\pi)^{-3/2} \times \int \exp\{i\mathbf{p}\mathbf{x} - iW_c(p)\,x_0\}\,a(\mathbf{p})\,d\mathbf{p}, \tag{24}$$

we obtain

$$H_{\text{int}} = ig\,(2\pi^3)^{-1/2} \times \int d\mathbf{p}\,d\mathfrak{f}\,f^{1/2}\,[a^*(\mathbf{p})\,a(\mathbf{p}+\mathfrak{f})\,\xi^*(\mathfrak{f})\exp\{-i\,[W_c(\mathbf{p}) - W_c(\mathbf{p}+\mathfrak{f}) - f]\,x_0\} - a^*(\mathbf{p})\,a(\mathbf{p}-f)\,\xi(\mathfrak{f})\exp\{-i\,[W_c(\mathbf{p}) - W_c(\mathbf{p}-\mathfrak{f}) + f]\,x_0\}], \tag{25}$$

where $f = |\,\mathfrak{f}\,|$, $\xi^*(\mathfrak{f})$ and $\xi(\mathfrak{f})$ are the Bose operators for creation and annihilation of a phonon with the quasimomentum $\mathfrak{f}$. The matrix elements (25) are well known, and carrying out the summation we obtain (evaluating Ω per unit volume):

$$\Omega = \Omega_0 + \Delta\Omega_1 + \Delta\Omega_2, \tag{26}$$

$$\Delta\Omega_1 = \frac{g^2}{4\pi^3}\int d\mathbf{p}\,d\mathfrak{f}\cdot f\,\frac{1-\exp\dfrac{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) + f}{kT}}{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) + f}\,\frac{\exp\dfrac{W_c(\mathbf{p}) - \mu}{kT}}{F(\mathbf{p},\mathfrak{f})}, \tag{27}$$

$$\Delta\Omega_2 = \frac{g^2}{4\pi^3}\int d\mathbf{p}\,d\mathfrak{f}\cdot f\,\frac{1-\exp\dfrac{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) - f}{kT}}{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) - f}\,\frac{\exp\dfrac{W_c(\mathbf{p}) - \mu + f}{kT}}{F(\mathbf{p},\mathfrak{f})}. \tag{28}$$

Here the function $F(\mathbf{p},\mathfrak{f})$ is determined by the relation

$$F(\mathbf{p},\mathfrak{f}) = \left[\exp\frac{W_c(\mathbf{p}+\mathfrak{f}) - \mu}{kT} + 1\right] \times \left[\exp\frac{W_c(\mathbf{p}) - \mu}{kT} + 1\right][e^{f/kT} - 1]. \tag{29}$$

A similar result is naturally obtained also for the phonon-hole gas.

Since $\Delta\Omega_1$ and $\Delta\Omega_2$ are obtained in the given approximation by adding together quantities corresponding to various electron states it is possible to introduce the thermodynamic potential $\Omega(\mathbf{p})$ for electrons having a momentum $\mathbf{p}$ and a momentum distribution function for the electrons $\Phi_c(\mathbf{p})$,

$$\Phi_c(\mathbf{p}) = -\,\partial\Omega(\mathbf{p})\,/\,\partial\mu. \tag{30}$$

From (26)–(29) we obtain

$$\Phi_c(\mathbf{p}) = \left[\exp\frac{W_c(\mathbf{p}) - \mu}{kT} + 1\right]^{-1} + \frac{g^2}{4\pi^3}\,\frac{\partial}{\partial(-\mu)} \times \int d\mathfrak{f}\cdot f F^{-1}(\mathbf{p}, f)\left\{\frac{1-\exp\dfrac{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) + f}{kT}}{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) + f} + \frac{1-\exp\dfrac{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) - f}{kT}}{W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p}) - f}\right\} \equiv \Phi_c^{(0)}(\mathbf{p}) + \Delta\Phi(\mathbf{p}). \tag{31}$$

In the general case the evaluation of the integral leads to quite awkward expressions. If we restrict ourselves to only the nondegenerate case we may write in place of (29)

$$F^{-1}(\mathbf{p},\mathfrak{f}) \approx \frac{\exp\{[2\mu - W_c(\mathbf{p}+\mathfrak{f}) - W_c(\mathbf{p})]\,/\,kT\}}{\exp(f\,/\,kT) - 1} \tag{32}$$

and integrate the resulting expression approximately, having in mind that because of the appearance of the factor $\exp\{-f^2/2m^* kT\}$ only small values of f are of importance. We then obtain:

$$\Delta\Phi(\mathrm{p}) = 8g^2m^*f_0 \exp\frac{\mu - W_c(\mathrm{p})}{kT}\left\{\frac{3}{2}e^{-z_0^2}\operatorname{ch}\alpha z_0 \right. \tag{33}$$

$$-\left(\frac{5}{2}+z_0^2+\frac{3}{4}\alpha^2\right)e^{-z_0^2}\frac{\operatorname{sh}\alpha z_0}{\alpha z_0}$$

$$-e^{-\alpha^2/4}\frac{\sqrt{\pi}}{8}\left(1+\frac{3}{2}\alpha^2\right)\left[\Phi\left(z_0-\frac{\alpha}{2}\right)\right.$$

$$\left.\left.+\Phi\left(z_0+\frac{\alpha}{2}\right)\right]\right\},$$

where

$$\Phi(x) = \frac{2}{\pi}\int_0^x e^{-t^2}\,dt; \tag{34}$$

$$z_0 = f_0/\sqrt{2m^*kT}; \quad \alpha = 2p/\sqrt{2m^*kT}.$$

In order of magnitude (at room temperature) $z_0 \sim 20$, and $\alpha \sim 1$ (for the most interesting values of p). Therefore (33) takes on the approximate form:

$$\Delta\Phi(p) \approx -2\sqrt{\pi}\,g^2m^*f_0\left(1+\frac{3p^2}{m^*kT}\right) \tag{35}$$

$$\times \exp\frac{\mu - W_0 - p^2/2m^*}{kT}\,\Phi(z_0).$$

Remembering that the principal term in the distribution function is in this case equal to $\exp\{[\mu - W_c(\mathrm{p})]/kT\}$, we find the electron density in the conduction band (whose statistical weight is $2G$);

$$N_c \approx 2G(2\pi)^{-3}(2\pi m^* kT)^{3/2}e^{(\mu - W_0)/kT} \tag{36}$$

$$\times\{1 - 6{,}9g^2m^*f_0\Phi(z_0)\}.$$

Thus, in the given approximation the influence of the phonon-electron interaction on the position of the level of the chemical potential reduces simply to a "renormaliztion" of the effective mass:

$$m^* \to m_r^* = m^*[1 - 6.9\,g^2m^*f_0\Phi(z_0)]^{2/3}. \tag{37}$$

The latter therefore becomes dependent on the temperature, increasing with increasing temperature. However, we wish to emphasize that for the calculation of various statistical quantities which characterize the semiconductor (for example the magnetic susceptibility, etc.) the "renormalization" alone is not sufficient— one should also use the improved distribution function.

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Dependence of the Life Time of Nonequilibrium Charge Carriers in Germanium on Temperature and Composition

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The analytic dependence of the position of the Fermi level on the temperature and on the composition of a semiconductor in the case of low trap density has been found in the form of approximate formulas. With the aid of these expressions an investigation is made of the dependence on the above factors of the lifetime of nonequilibrium charge carriers in a semiconductor for small deviations from thermodynamic equilibrium which permits one to improve the formula for the determination of the binding energy of electrons and of holes in the traps.

1. INTRODUCTION

THE lifetime τ of the nonequilibrium charge carriers is one of the fundamental electrophysical characteristics of a semiconductor. For a low density of recombination centers (traps) and for small deviations from thermodynamic equilibrium, τ is expressed by the formula of Shockley and Read[1]

$$\tau = \tau_{p0}(n_0 + n_1)/(n_0 + p_0) + \tau_{n0}(p_0 + p_1)/(p_0 + n_0). \tag{1}$$

Here $n_0 = N_2 \exp\{-(E_u - F)/kT\}$, $p_0 = N_1 \exp\{-(F - E_l)/kT\}$ are the electron and hole densities in thermodynamic equilibrium;

$$n_1 = N_2 \exp\{-(E_u - E_t)/kT\}, \quad p_1 = N_1 \exp\{-(E_t - E_l)/kT\} \tag{2}$$

are quantities which have the dimensions of density and which are numerically equal to the equilibrium electron and hole densities when the Fermi level F coincides with the energy levels of the traps;

$$\tau_{p0} = 1/A_h N_t, \quad \tau_{n0} = 1/A_e N_t, \tag{3}$$

where N_t is the density of traps, A_e and A_h are respectively the probabilities of capture by the trap of an electron and of a hole.

The dependence of the lifetime, and also of other electrophysical characteristics (conductivity, Hall coefficient) on the temperature and on the composition of the semiconductor is expressed not only explicitly through T, E_u, E_l, E_t, A_e, A_h, etc., but also implicitly through the Fermi level F. Consequently it is first of all necessary to determine the law for the variation with temperature of the position of the Fermi level in semiconductors of different compositions. We shall take a definite composition of the semiconductor to mean that we are given the properties of the basic material, and also the densities and the characteristics of all the impurities and structural defects which it contains. In contrast to Refs. 2–5*, where the problem of the Fermi level is investigated by special graphical methods which require the construction for each new composition of its own family of curves, the present article gives an analytic solution of the problem and presents the result in the form of sufficiently simple approximate expressions useful for calculations in specific cases.

2. CALCULATION OF THE POSITION OF THE FERMI LEVEL

The position of the Fermi level under conditions of thermodynamic equilibrium is obtained from the normalization condition

$$\int_{-\infty}^{\infty} N(E) f(E)\, dE = N_0, \qquad f(E) = [e^{(E-F)/kT} + 1]^{-1}, \tag{4}$$

where $N(E)$ is the level density , and N_0 is the total electron density at a given point of the semiconductor.

Assuming that in the forbidden band there exist in the general case local levels of S different types with binding energies equal respectively to E_i ($i = 1,2,3, \ldots, S$) for which $N(E) = N_i\,\delta(E - E_i)$, we obtain from (4)

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*The formulas for F, obtained in Refs. 5-8, refer to a very special case which is realized, as is shown below, only within a very small range of temperatures.

$$\int\limits_{1} N(E)\, f(E)\, dE + \int\limits_{2} N(E)\, f(E)\, dE + \sum_{1}^{S} N_i\, f(E_i) = N_0, \qquad (5)$$

where the integrals 1 and 2 are taken respectively over the lower and the upper bands. In terms of densities, Eq. (5) takes the form:

$$n + N_{10} - p + \sum_{i=1}^{S} n_i = N_0, \qquad (6)$$

where N_{10} is the total number of levels in the lower band.

If of the S types of local levels S_1 are of the donor type (i.e., they are occupied by electrons in the electrically neutral condition), then

$$N_0 = N_{10} + \sum_{i=1}^{S_1} N_i - \rho/q, \qquad (7)$$

where N_i is the density of donor levels of the i th type, and ρ is the electric charge density at the given point. Substituting (7) into (6) and restricting ourselves to the electrically neutral case we obtain the equation

$$n + \sum_{1}^{S} n_i - p = \sum_{1}^{S_1} N_i, \qquad (8)$$

which determines F.

For non-degenerate semiconductors defined by the conditions

$$(E_u - F)/kT \gg 1, \ (F - E_l)/kT \gg 1, \qquad (9)$$

the distribution of electrons in the upper band and the distribution of holes in the lower band correspond to Boltzmann statistics

$$n = \int\limits_{E_u}^{\infty} N(E)\, f(E)\, dE \approx \int\limits_{F_u}^{\infty} N(E) \exp\{-(E - F)/kT\}\, dE = N_2 \exp\{-(E_u - F)/kT\},$$

$$p = \int\limits_{-\infty}^{E_l} N(E)\, f(E)\, dE \approx \int\limits_{-\infty}^{E_l} N(E)\, e^{-(F-E)/kT}\, dE = N_1 e^{-(F-E_l)/kT}. \qquad (10)$$

Here E_l and E_u are respectively the top of the lower band and the bottom of the upper band (Fig. 1);

$$N_1 = \int\limits_{-\infty}^{E_l} N(E) \exp\{-(E_l - E)/kT\}\, dE = 2\,(2\pi m_h kT/h^2)^{3/2},$$

$$N_2 = \int\limits_{F_u}^{\infty} N(E) \exp\{-(E - E_u)/kT\}\, dE = 2\,(2\pi m_e kT/h^2)^{3/2} \qquad (11)$$

are the reduced values of the level densities in the bands; m_e and m_h are the effective masses of the electron and of the hole.

Selecting for the zero of energies the position of the Fermi level in a pure semiconductor (see Fig. 1)

$$F_0 = {}^1/_2\,(E_u + E_l) + {}^1/_2 kT \ln(N_1/N_2) = {}^1/_2\,(E_u + E_l) + {}^3/_4\, kT \ln(m_h/m_e), \qquad (12)$$

which is determined by the condition $n=p$ (all $N_i = 0$) and introducing the notation

$$\varphi = (F - F_0)/kT_0;\ \varepsilon_l = (E_l - F_0)/kT_0;$$

$$\varepsilon_u = (E_u - F_0)kT_0;$$

$$\varepsilon_i = (E_i - F_0)/kT_0;\ \alpha_u = e^{\varepsilon_u/\Theta};\ \alpha_l = e^{\varepsilon_l/\Theta};$$

$$\alpha_i = e^{\epsilon_i/\Theta};\ y = e^{\varphi/\Theta};\ \nu_i = N_i/N_0;\ \ \Theta = T/T_0;$$

$$\varkappa = (N_2/N_1)^{1/2} = (m_e/m_h)^{3/4};$$

$$N_0 = \sqrt{N_1(T_0)\, N_2(T_0)}; \qquad (13)$$

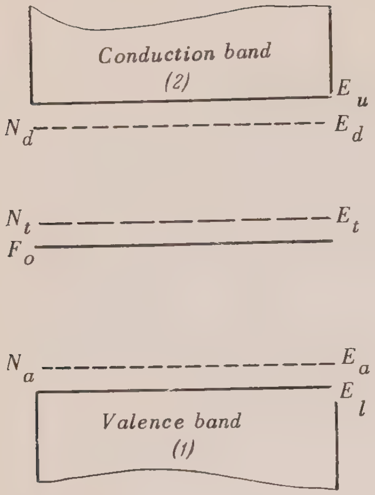

FIG. 1. Energy diagram of a semiconductor.

$$\mathcal{E}_i=E_i-F_0;\ \mathcal{E}_l=E_l-F_0;\ \mathcal{E}_u=E_u-F_0, \quad (14)$$

where $T_0 = 300°$ K we shall bring Eq. (8) to the following form:

$$\varkappa\Theta^{3/2}\frac{y}{\alpha_u}+\sum_1^S\frac{\nu_i}{1+\alpha_i/y}-\frac{1}{\varkappa}\Theta^{3/2}\alpha_l/y=\sum_1^{S_1}\nu_i. \quad (15)$$

In the general case this equation is of order $S+2$ with respect to y. We note, however, that

$$\frac{\alpha_i}{y}\begin{cases}\gg 1 & \text{for } (E_i-F)/kT\gg 1,\\ \ll 1 & \text{for } (F-E_i)/kT\ll 1.\end{cases} \quad (16)$$

Consequently, if the difference in the binding energies of the electrons on local levels of different types is much larger than kT, then (15) represents an equation of order not higher than the third.

The quantity y introduced in place of the Fermi level has a simple physical meaning

$$y=e^{(F-F_0)/kT} \quad (17)$$

$$=(N_2/N_1)^{1/2}\exp\{(2F-E_u-E_l)/2kT\}=\sqrt{n/p}.$$

For a semiconductor whose energy diagram is given in Fig. 1 equation (15) may be written as follows:

$$\varkappa\Theta^{3/2}\frac{y}{\alpha_u}+\frac{\nu_a}{1+\alpha_a/y}-\frac{\nu_d}{1+y/\alpha_d}-\frac{1}{\varkappa}\Theta^{3/2}\frac{\alpha_l}{y}=0. \quad (18)$$

The number of electrons in traps is assumed to be negligibly small because of the low density of traps.

In the following we shall restrict ourselves to an investigation of semiconductors with a small activation energy for both donors and acceptors. A typical example of such semiconductors is germanium^8:

$$|E_u-E_d|\sim 0.04 \text{ ev}, \quad |E_a-E_l|\sim 0.04 \text{ ev}, \quad (19)$$

for which all the following calculations have been carried out.

Let us consider the n-type of germanium $\nu_d-\nu_a>0$. Since the position of the Fermi level in pure germanium has been chosen as the zero of energy, the Fermi level in the case under investigation is positive at all temperatures ($\varphi \geq 0$) and tends to zero for $\Theta \gg 1$.

Let us first of all show that in n-type germanium one may take $\alpha_a/y \ll 1$ at all experimental temperatures, i. e., one may neglect the density of holes on acceptor levels. The ratio α_a/y is determined by

$$\alpha_a/y=\exp\{(\varepsilon_a-\varphi)/\Theta\}<\exp\{\varepsilon_a/\Theta\} \quad (20)$$
$$=(m_e/m_h)^{3/4}\exp\{[E_a-{}^1/_2(E_u+E_l)]/kT\}.$$

The distance of acceptor levels from the middle of the band is ~ 0.3 ev; consequently α_a/y is certainly less than 0.01 up to temperatures $T\approx 500°$ C.

We shall denote by the term low temperature region that range of temperatures within which one can neglect the density of holes in the valence band in comparison with the density of electrons in the conduction band. We shall determine the upper limit Θ_I of the temperatures in this region from the condition $p/n=0.01$, i.e., $y=10$ [see (17)].

In the low temperature region Eq. (18) takes on the following form:

$$\varkappa\Theta^{3/2}\frac{y}{\alpha_u}+\nu_a-\frac{\nu_d}{(y/\alpha_d)+1}=0. \quad (21)$$

Its solution is

$$y=(\alpha_u/2\varkappa\Theta^{3/2})\{[(\nu_a+\varkappa\Theta^{3/2}\alpha_d/\alpha_u)^2 \quad (22)$$
$$+4\varkappa(\alpha_d/\alpha_u)\Theta^{3/2}(\nu_d-\nu_a)]^{1/2}-(\nu_a+\varkappa\Theta^{3/2}\alpha_d/\alpha_u)\}.$$

The low temperature region $0 \leq \Theta < \Theta_I$ may be divided into three temperature intervals in which different approximate expressions for y are possible.

Near absolute zero where one may neglect in (21) the density of free electrons $\nu = \varkappa\Theta^{3/2} y/\alpha_u$,

$$y = \alpha_d \frac{\nu_d - \nu_a}{\nu_a}\left\{1 - \frac{\varkappa}{4(\nu_d - \nu_a)}\frac{\alpha_d}{\alpha_u}\Theta^{3/2}\right. \tag{23}$$

$$\left.\times\left[4\left(\frac{\nu_d}{\nu_a} - \frac{1}{2}\right)^2 - 1\right]\right\} \approx \alpha_d \frac{\nu_d - \nu_a}{\nu_a}.$$

The boundary of this temperature interval Θ_1 is determined from the equation

$$\Theta_1^{3/2} \exp\{-(\varepsilon_u - \varepsilon_d)/\Theta_1\} \tag{24}$$

$$= \begin{cases} 0.01\,\nu_a^2/(\nu_d - \nu_a)\varkappa, & \text{if} \quad \nu_d > 2\nu_a, \\ 0.01\,\nu_a/\varkappa, & \text{if} \quad \nu_d < 2\nu_a. \end{cases}$$

Let us now assume in Eq. (21)

$$\nu_d/(y/\alpha_d + 1) \approx \nu_d(1 - y/\alpha_d), \tag{25}$$

which corresponds to assuming a low degree of occupancy of donor levels. In such a case,

$$y = \alpha_d(\nu_d - \nu_a)/(\nu_d + \varkappa\Theta^{3/2}\alpha_d/\alpha_u). \tag{26}$$

Turning to Eq. (18) and demanding that the error introduced by the approximation into the free electron density should not exceed 1% we obtain the formula

$$(\nu_d - \nu_a)/2\varkappa\Theta_2^{3/2}\alpha_d/\alpha_u = 0.01, \tag{27}$$

which determines the lower boundary of the temperatures in the interval under consideration.

Now substituting $y = 10$ into expression (26) we determine Θ_I – the common upper limit of the whole low temperature region:

$$\alpha_d \frac{\nu_d - \nu_a}{\nu_d + \varkappa\Theta_I^{3/2}(\alpha\ /\alpha_u)} \approx \alpha_u(\nu_d - \nu_a)/\varkappa\Theta_I^{3/2} = 10. \tag{28}$$

ν_d has been neglected in the denominator because already at $\Theta = 0.4$ N_0 $\varkappa\Theta^{3/2}\alpha_d/\alpha_u \gg 10^{17}\text{cm}^{-3}$, and from (28) it follows that $\Theta_I \sim 1$. Consequently at the usual impurity densities Θ_I does not depend on α_d, i.e., on the magnitude of the binding energy of the electron on impurity levels.

We see that $\Theta_2 < \Theta_I$ in all cases for actually occurring values of the binding energy of the electron on traps. Consequently there exists a finite temperature interval in the low temperature region in which expression (22) for y can be approximated by the expression (26). A more exact form for y in this interval is given by the following:

$$y = \{\alpha_d(\nu_d - \nu_a)/(\nu_d + \varkappa\Theta^{3/2}\alpha_d/\alpha_u)\}$$

$$\{1 + (\nu_d - \nu_a)/(4\varkappa\Theta^{3/2}\alpha_d/\alpha_u)\}. \tag{29}$$

At temperatures $\Theta_1 < \Theta < \Theta_2$ the general formula (22) should be used.

We now proceed to investigate the high temperature region, i.e., the region $\Theta \geq \Theta_I$. In doing so we must take into account in Eq. (18) the density of holes in the valence band.

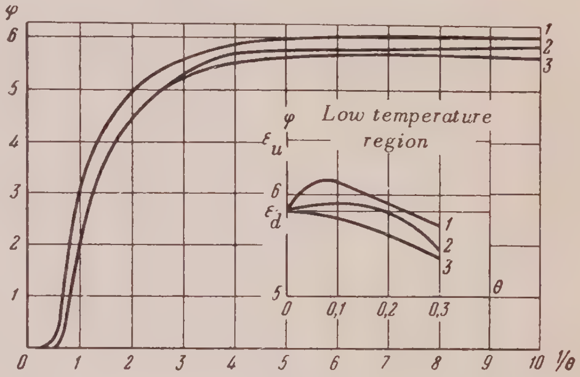

FIG. 2. Dependence of the position of the Fermi level on the temperature. 1– $N_d = 7.5\times10^{15}$ cm^{-3}, $N_a = 7.5\times10^{10}$ cm^{-3} ; 2– $N_d = 7.5\times10^{14}$ cm^{-3}, $N_a = 7.5\times10^{13}$ cm^{-3} ; 3– $N_d = 7.5\times10^{15}$cm^{-3}, $N_a = 6.75\times10^{15}$cm^{-3}

As a consequence of the fact that $\Theta_I > \Theta_2$, Eq. (18) can be written as follows:

$$\varkappa\Theta^{3/2}(y/\alpha_u) + \nu_d(y/\alpha_d) - \Theta^{3/2}\alpha_l/\varkappa y = \nu_d - \nu_a. \tag{30}$$

Its solution has the following form:

$$y = \frac{\alpha_u}{2\varkappa\Theta^{3/2}}\left\{\left[4\Theta^3\frac{\alpha_l}{\alpha_u} + (\nu_d - \nu_a)^2\right]^{1/2} + (\nu_d - \nu_a)\right\}, \tag{31}$$

where we have taken into account that for $\Theta \geq \Theta_I$, $\varkappa\Theta^{3/2}\alpha_d/\alpha_u \gg \nu_d$. For $\Theta = \Theta_I$ Eq. (31) is the same as (26).

In the high temperature region we shall pick out two temperatures: Θ_3, which corresponds to the impurity conductivity being equal to the intrinsic conductivity [$\nu = 2(\nu_d - \nu_a)$], and Θ_4, which corresponds to the intrinsic conductivity [$\nu \approx 100(\nu_d - \nu_a)$]; Θ_3 and Θ_4 are determined by the formulas:

$$\Theta_3^3 \alpha_l / \alpha_u = 2(\nu_d - \nu_a)^2,$$

$$[(\nu_d - \nu_a)/2\Theta_4^{3/2}] \exp\{\varepsilon_u/\Theta_4\} = 0.01. \quad (32)$$

These temperatures characterize the transition from the impurity to the intrinsic conductivity.

Figures 2 and 3 show the result of the calculation of the position of the Fermi level and of the density ν according to the above formulas. The boundaries of the temperature intervals are shown there also.

For a p-type semiconductor all the results obtained above also hold, only N_d and N_a should be interchanged. $\mathcal{E}_u$ should be replaced by $-\mathcal{E}_l$, $-\mathcal{E}_l$ by $\mathcal{E}_u$, f by $-f$, $\mathcal{E}_a$ by $-\mathcal{E}_d$, $\mathcal{E}_d$ by $-\mathcal{E}_a$, and $\varkappa$ by $1/\varkappa$.

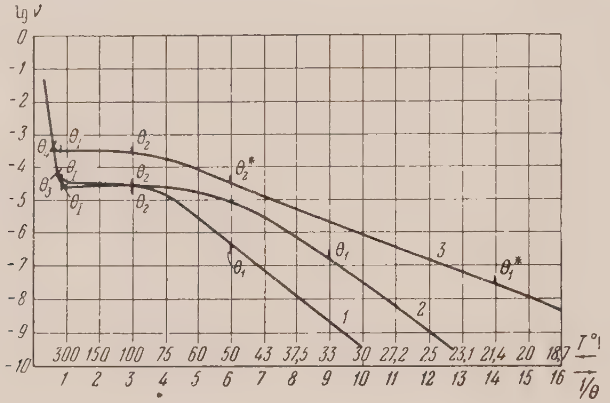

FIG. 3. Temperature dependence of the density of charge carriers. 1– $N_d = 7.5 \times 10^{15}\,\text{cm}^{-3}$, $N_a = 6.75 \times 10^{15}\,\text{cm}^{-3}$; 2–$N_d = 7.5 \times 10^{14}\,\text{cm}^{-3}$, $N_a = 7.5 \times 10^{13}\,\text{cm}^{-3}$; 3– $N_d = 7.5 \times 10^{15}\,\text{cm}^{-3}$, $N_a = 6.75 \times 10^{10}\,\text{cm}^{-3}$.

In concluding this section we note that the direct use of the formula for extrinsic semiconductors [5,8,10,11]

$$\sigma = A \exp\{-(E_u - E_d)/2kT\} \quad (33)$$

as applied to germanium may lead to quite incorrect values of the ionization levels of the impurities $E_u - E_d$ in the interpretation of experimental data. Formula (33) follows from (21) if the term ν_a is neglected, and if $y/\alpha_d \gg 1$, which leads to.

$$y = \sqrt{\nu_d/\varkappa}\,\Theta^{-3/4} \exp\{(\varepsilon_u + \varepsilon_d)/2\Theta\};$$

$$\varphi = \Theta \ln y \approx {}^1\!/_2(\varepsilon_d + \varepsilon_u). \quad (34)$$

Conditions under which these neglections will hold with an accuracy to $\delta\%$ for concentrations

$$100/\delta \leqslant y/\alpha_d \leqslant 0.01\delta\nu_d/\nu_a \quad (35)$$

show that (33) may hold only in the case of very strongly extrinsic germanium: $\nu_d/\nu_a \gg 10^4/\delta^2$, i.e., at least for $\nu_d \sim 10^3\,\nu_a$.

Moreover, it follows from (35) that in this case also, the approximation (34), which leads to (33), holds only within a very limited temperature interval. For $n_d = 7.5 \times 10^{15}\,\text{cm}^{-3}$, $N_a = 7.5 \times 10^{12}\,\text{cm}^{-3}$ and $\delta = 10\%$ the region in which (33) is applicable is limited to the interval $34°\,\text{K} \leq T \leq 48°\,\text{K}$.

3. LIFETIME

Introducing dimensionless quantities (13) and adopting as the unit of time τ_{p0} ($\tau^* = \tau/\tau_{p0}$), we can reduce formula (1) to the following form:

$$\tau^{*} = \frac{(1+\alpha_t/y)\,(1+\gamma/\alpha_t\nu)}{1+y^{-2}}, \tag{36}$$

where

$$\gamma = \frac{\tau_{n0}}{\tau_{p0}};\ \alpha_t = \exp\{\varepsilon_t/\Theta\}.$$

The system of equations (36) and (18) represents a parametric expression of the dependence of the lifetime on the temperature and on the composition of a semiconductor. In the low temperature region ($\Theta \leq \Theta_1$) in place of (18) one should use (22), while in the high temperature region ($\Theta \geq \Theta_1$) one should use formula (31).

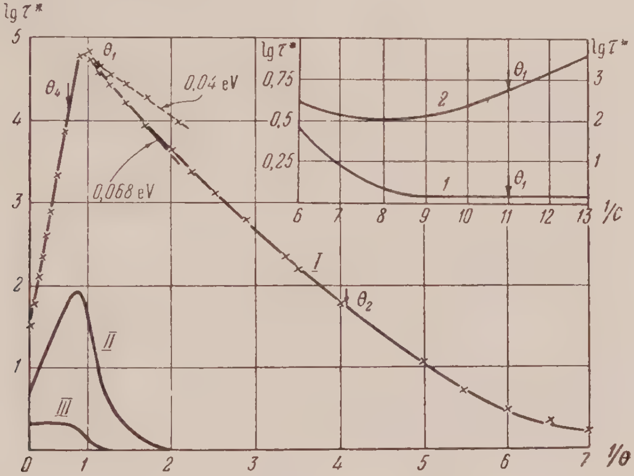

FIG. 4. Temperature dependence of the lifetime for different positions of the traps.

Figure 4 shows the result of eliminating y from the system of equations (36) and (18) for $N_a = 10^{13}$ cm^{-3}, $N_d = 10^{14}$ cm^{-3}, $\varkappa=1$, $\gamma=1$ and for the other parameters determined by (19). The curve *I* shows the dependence of τ^* on the temperature in the case that the recombination levels are situated at the same height as the donors $\mathcal{E}_u - \mathcal{E}_t = \mathcal{E}_u - \mathcal{E}_d = 0.04$ ev. Curve *II* corresponds to $\mathcal{E}_u - \mathcal{E}_t = 0.22$ ev, while curve *III* corresponds to $\mathcal{E}_u - \mathcal{E}_t = 0.375$ ev. In the latter case the trap levels are situated in the middle of the forbidden band.*

We have shown separately the behavior of lg τ^* at temperatures close to absolute zero ($\Theta \leq \Theta_1$). In addition to the continuation of the graph for $\mathcal{E}_u - \mathcal{E}_t = 0.04$ ev (curve 1, left hand scale) we have also shown here the graph for the case when the trap levels are situated closer to the conduction band than the donor levels (curve 2, right hand scale, which corresponds to $\mathcal{E}_u - \mathcal{E}_t = 0.02$ ev).

As may be seen from Fig. 4, the lifetime of nonequilibrium carriers does not vary monotonically with temperature. As the temperature is increased τ^* increases at first, then reaches a maximum, then decreases. If $|\epsilon_t| > \epsilon_d$, then τ^* increases also as $\Theta \rightarrow 0$ (see curve 2). An exception occurs in the case when the recombination levels are situated in the middle of the forbidden band (more cccurately when $\epsilon_t = 0$, i.e., when the traps coincide with the position of the Fermi level in pure germanium). For such a position of the traps τ^* increases monotonically, and then stays at a constant value (see curve *III*).

Let us elucidate the physical meaning of the results obtained above. The relaxation of the nonequilibrium electron density is determined by two processes :1) the recombination of excess electrons with equilibrium empty traps, and 2) the recombination of equilibrium electrons with excess empty traps. For small deviations from thermodynamic equilibrium we can neglect the quadratic effect—the recombination of excess electrons with excess traps. The mathematical formulation of the above appears as follows:

$$-dn'/dt = A_e n' N_{et\,n} + (A_e n + B_e)\, N'_{et\,n}, \tag{37}$$

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*We note that $\varkappa=1$, i.e., the middle of the forbidden band coincides with the position of the Fermi level in pure germanium.

where A_e and B_e are respectively the probabilties per unit time for the recombination and the thermal release of an electron, n and n' are the equilibrium and excess electron densities, N_{et} and N'_{et} are the equilibrium and excess empty trap densities. The second term on the right hand side of (37) contains the expression $B_e N'_{et}$ because the appearance of the excess density of empty traps N'_{et} not only speeds up the recombination of electrons, but also slows down their ejection from the traps back into the conduction band. In a similar way, for the hole we have

$$-dp'/dt = A_h p' N_{ft^3} + (A_h p + B_h) N'_{ft} \cdot \qquad (38)$$

Evidently the excess density of filled traps is given by

$$N'_{ft^3} = -N'_{et}\,n. \qquad (39)$$

Under the conditions of stationary pair formation and of low trap density under which formula (1) was obtained, the lifetime of nonequilibrium carriers represents essentially the average time for the disappearance of an electron-hole pair, and not for the disappearance of either one of these two particles. The act of the recombination of the electron-hole pair is made up of two elementary transitions 1) of an electron into an empty trap and 2) of a hole into an occupied trap. As a result of such a process the electron-hole pair disappears, while the degree of occupancy of the traps is not changed. The lifetime is determined fundamentally by the slower one of the two transitions. However, a simple addition of the characteristic times of the elementary acts cannot be used, as this would correspond to such conditions under which the recombination of one of the particles, for example a hole, would become possible only after the recombination of the other particle, for example, an electron, had already occurred.

If the characteristic times of the elementary transitions of an electron and of a hole are quantities of different orders of magnitude, then the lifetime τ is equal to the larger one of these two characteristic times. In the general case the value of the lifetime of the electron-hole pair is influenced by the fact that the transitions of both particles to traps may occur not only one following another, but also simultaneously. Instead of a singlevalued dependence, a correlation exists between the transitions (the transition of the electron may occur either simultaneously with or after the transition of the hole, etc.). For the stationary case, when the rate of recombination is equal to the rate of pair formation (illumination or injection) this correlation finds its expression in Eqs. (1) and (36).

We now proceed to the examination of individual specific cases. We shall restrict ourselves to an investigation of n-type germanium. The results which we shall obtain can be easily adapted also to the case of p-type germanium.

EXAMPLE 1. The traps are situated in the upper half of the forbidden band below the donors (Fig. 5a). Near absolute zero all the traps are filled, and the lifetime of the hole is equal to τ_{p0} or to unity in dimensionless units. Since the Fermi level $\varphi \approx \epsilon_d$, i.e., since it is close to the conduction band, the trap which becomes vacant after the recombination of a hole is filled almost immediately (in a time smaller than τ_{p0}) by some electrons. The lifetime is determined by the recombination time of the hole, i.e., $\tau^* = 1$. As the temperature is increased, a partial emptying of the traps takes place. The density of filled traps is decreased, and consequently the probability of recombination for holes is also decreased. Consequently the lifetime is increased. An increase in the temperature leads to an increase in the free electron density and for $\epsilon_t > 0$ to a considerable increase in the number of vacant traps. Therefore the characteristic time for an elementary electron transition is very small and has no effect on the value of τ.

Calculations using formulas (36) and (22) show that over a sufficiently wide interval to the right of the maximum

$$\tau^* = [\varkappa\Theta^{3/2}/(\nu_d - \nu_a)] \exp\{-(\varepsilon_u - \varepsilon_t)\,\Theta\}. \qquad (40)$$

From (40) it follows that

$$d \ln \tau^*/d(1/\Theta) = -(\varepsilon_u - \varepsilon_t) - {}^3/{}_2\Theta. \qquad (41)$$

Under the condition that

$$(\varepsilon_u - \varepsilon_t)/\Theta \gg {}^3/{}_2 \qquad (42)$$

the slope of the curve of the dependence of the lifetime on the temperature plotted in terms of $1/\Theta$, $\ln \tau^*$ is simply equal to the distance of the

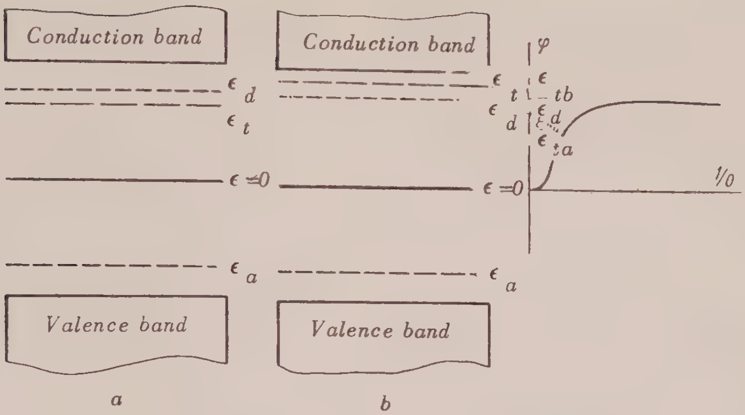

FIG. 5. Energy diagram of *n*-type germanium for $\epsilon_t > 0$. On the right-hand side of the diagram is shown the dependence on the temperature of the Fermi level, with the energies of the traps in both cases *a* and *b* indicated on the vertical axis.

recombination levels from the bottom of the conduction band. This gives the possibility of a direct experimental determination of the binding energy of the electron on recombination levels.

It should be noted that such a method of determining $\epsilon_u - \epsilon_t$ is by no means always possible . Indeed, in (42) one may take $\Theta_I \sim 1$ ($T \sim 300°$ K), since the temperature range under investigation is adjacent to the temperature Θ_1 , beyond which an appreciable intrinsic conductivity appears in germanium which then leads to a decrease of τ^* . Noting that the unit of energy is $kT_0 = 0.026$ ev, we obtain from (42)

$$\Delta \mathcal{E}_e = \mathcal{E}_u - \mathcal{E}_t \geqslant 0.04 \text{ ev.} \tag{43}$$

Consequently, only in the case when the traps are situated in the immediate vicinity of the middle of the forbidden band ($\Delta \mathcal{E}_e \sim 0.37$ ev) may the binding energy of an electron on a trap be determined sufficiently accurately from the slope of the curve of ln τ plotted against $1/T$. In the general case the neglect of the term 3/2 Θ in (41) may lead to a considerable error in the determination of $\Delta \mathcal{E}_e$.

Let us now turn to Fig. 4. The slope of the curve *I* to the right of Θ_1 is equal to 1:19. Going over to natural logarithms and to dimensional units we find that the slope of the curve corresponds to 0.068 and not to 0.04 ev. If one determines the electron binding energy from the slope of the curve in this case one will make an error of 72%.

In order to determine the position of the traps from the experimental data on the lifetime one should construct in the region of the density plateau the curve showing the dependence of ln $(\tau T^{-3/2})$ on $1/T$.

As may be seen from (40), at temperatures to the right of Θ_1 ($\Theta < \Theta_1$), the following exact equality holds:

$$\frac{d \ln(\tau^*/\Theta^{3/2})}{d(1/\Theta)} = -(\varepsilon_u - \varepsilon_t) \text{ for } \frac{d \ln(\tau/T^{3/2})}{d(1/T)} = -(\mathcal{E}_u - \mathcal{E}_t)/k. \tag{44}$$

The corresponding curve is shown dotted in Fig. 4. Its slope in dimensional units is equal to 0.04 ev, which exactly corresponds to the position of the traps.

The considerations outlined above have not been taken into account by Hall.[12] If one introduces the necessary correction for $\Theta^{3/2}$, then the position of the traps determined from Hall's experimental data turns out to be equal to 0.22 and not to 0.18 ev, which corresponds to an error of 20% committed by Hall.

At temperatures higher than Θ_I , the density of thermally generated(equilibrium) holes first appears and rapidly increases.. If for $\Theta < \Theta_I$ the lifetime was determined largely by the process of the recombination of excess holes with the equilibrium filled traps

$$-dp'/dt \approx A_1^h p' N_{ft^3} \tag{45}$$

[see (38)] , then for $\Theta > \Theta_1$ the process of the recombination of equilibrium holes with excess filled traps also begins to play an essential role. As long as the lifetime was determined by the first term in (38), it increased wihh increasing temperature, because the probability of recombination correspondingly decreased due to a decrease of N_{ft} In the region of high temperatures, in which the second term of Eq. (38) predominates, τ decreases with temperature, because the probability of recombination increases with increasing p. This explains the non-monotonic character and the maximum of the curves giving the temperature dependence of τ^*.

The above investigation shows that for $\epsilon_t > 0$ the lifetime of the electron-hole pair is fundamentally determined by the time of a transition to a trap of that one of the components of the pair which belongs to the minority current carriers in germanium of a given conductivity type. An increase in τ by several orders of magnitude in comparison with τ_{p0}* at intermediate temperatures is determined by the fact that in this temperature range the degree of occupancy of traps is small, i.e., the probability of capture of a hole is small. For $\Theta \to 0, N_{ft} \to N_t$, and for $\Theta \to \infty$, $N_{ft} \to 1/2\, N_t$, as a result of which τ becomes of the order of τ_{p0} (or τ_{n0}).

If $\epsilon_t = 0$, then in n-type germanium ($y > 1$) we have for all Θ

$$\tfrac{1}{2}N_t \leqslant N_{t3} \leqslant N_t. \qquad (46)$$

The occupancy of traps remains quite high at any temperature and therefore τ^* is small (see curve *III*, Fig. 4). Moreover, in the region of intrinsic conductivity ($y \approx 1$) $N_{ft} = 1/2\, N_t$. Therefore for $\epsilon_t = 0$ the dependence of $\ln \tau^*$ on $1/\Theta$, instead of having a maximum and a falling off at high temperatures, acquires a plateau.

EXAMPLE 2. The traps are situated above the donors (Fig. 5b). The temperature dependence of τ^* differs from the one discussed above only for $\Theta < \Theta_1$, where a decrease of temperature is accompanied by an increase of τ^* (see curve 2 in Fig. 4). The reason for this is the following.

If $\epsilon_t > \epsilon_d$, then for $\Theta = 0$ all the traps are vacant, since $\varphi \to \epsilon_d$ as $\Theta \to 0$. Therefore the probability of hole capture at $\Theta = 0$ is equal to zero and correspondingly $\tau = \infty$. In this case as the temperature is raised the degree to which the traps are filled at first increases, and only then begins to decrease, while for $\epsilon_t < \epsilon_d$ N_{ft} decreases beginning with $\Theta = 0$. For $\epsilon_t > \epsilon_d$, in the temperature range up to Θ_1, N_{ft} increases with increasing temperature and consequently τ decreases.

For $\epsilon_t = \epsilon_d$ the degree to which the traps are filled remains practically constant up to $\sim \Theta_1$, as a result of which

$$\tau^* = \tau_h/\tau_{p0} = \nu_d/(\nu_d - \nu_a) = \text{const.} \qquad (47)$$

The ratio $(\nu_d - \nu_a)/\nu_d$ is equal to the degree to which the donors are filled, and consequently to the degree to which the traps are filled which have the same energy as the donors.

EXAMPLE 3. The traps are situated in the lower half of the forbidden band (Fig. 6). The character of the temperature dependence is qualitatively the same as for $\epsilon_t > 0$. An analysis of the formulas shows that for $\gamma = 1$, τ depends only on $|\epsilon_t|$

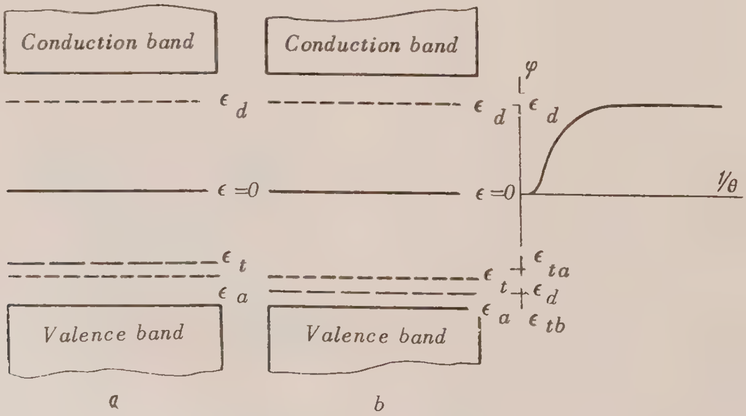

FIG. 6. Energy diagram for n-type germanium for $\epsilon_t < 0$.

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*By comparison with τ_{n0} in p-type germanium.

If $-\epsilon_t < \epsilon_d$ (Fig. 6a) then τ^* increases monotonically in the low temperature region, then goes through a maximum, and falls off in the high temperature region. The general form of the function $\tau^*(\Theta)$ is the same as that shown in Fig. 4 (curves *I* and *II*). However, the physical character of the processes which determine the temperature dependence of τ^* is quite different. This may be seen already from the fact that the temperature dependence is contained in terms which include as a factor γ, i.e., the lifetime is determined not by the process of recombination of holes (minority carriers), but by the process of recombination of electrons (majority carriers),

As may be seen from Fig. 6 for $\epsilon_t < 0$ the occupancy of traps in *n* -type germanium ($\varphi > 0$) remains at all actual temperatures close to unity, and only for $\Theta > \Theta_4$ it tends in the limit to one-half. Therefore, practically at all temperatures the capture of a hole by a trap occurs during a time τ_{p0}. On the other hand, the transition of an electron into a trap requires a considerably longer time than τ_{n0}, since the traps are almost entirely occupied by electrons.

In accordance with the above, the dominant role in Eq. (37) is played by the second term on the right-hand side; in other words the rate at which the process of disappearance of pairs takes place is determined by the recombination of equilibrium electrons *n* with excess empty traps N'_{et} ;

$$-dn'/dt \approx A_e n N'_{et}. \qquad (48)$$

As *n* increases, the lifetime τ decreases, while when N'_{et} is increased τ decreased.

The temperature dependence of τ is determined by which of these two factors dominates.

In the temperature interval from Θ_2 to Θ_1, which corresponds to the saturation of impurity conductivity , $n \approx$ const, while N'_{et} will decrease as the temperature increases, because of the thermal ejection of electrons from the valence band into empty traps. In other words, here the principal role is played by the thermal barrier,[13] which oonsists of the fact that a hole captured by a trap is ejected back into the valence band before an electron from the conduction band has time to enter this trap. As a result τ in this case increases with temperature.

As the temperature goes through Θ_1 an appreciable density of equilibrium holes appears both in the valence band and in the traps which leads to a decrease in τ just as in the case $\epsilon_t > 0$ which was considered in detail above.

The principal difference of the physical processes taking place in *n*-type germanium for $\epsilon_t > 0$ and $\epsilon_t < 0$ consists of the following. For $\epsilon_t > 0$ the lifetime is determined primarily by the characteristic time of the elementary act of the transition of a hole into a trap. At intermediate temperatures this may exceed τ_{p0} by a large factor as a result of the fact that a large fraction of the traps is vacant which makes the recombination of the minority carriers more difficult. For $\epsilon_t < 0$, the lifetime is determined mainly by the characteristic time of the elementary act of the transition of an electron into a trap. At intermediate temperatures this may exceed τ_{n0} by a large factor because of the fact that a thermal barrier is created at the traps: the vacant traps are filled by electrons from the valence band which makes the recombination of conduction electrons (majority carriers) more difficult.

In the general case the lifetime is determined primarily by the characteristic time of the elementary transitions of the minority carriers when the traps are situated in the same half of the forbidden band as the levels of the dominant impurity ($\epsilon_t > 0$ in *n*-type germanium; $\epsilon_t < 0$ in *p*-type germanium). The lifetime is determined primarily by the characteristic time of the elementary transitions of the majority carriers when the traps are situated in the other half of the forbidden band ($\epsilon_t < 0$ in *n*-type germanium, $\epsilon_t > 0$ in *p*-type germanium).

The further are the recombination levels situated from the middle of the forbidden band, the greater is the role played by the effect of the emptying of the traps , or by the thermal barrier , and therefore the larger will be the value of the lifetime τ at intermediate temperatures in comparison with τ_{p0} or τ_{n0}. This can be seen from Fig. 4 where for $\mathcal{E}_u - \mathcal{E}_t = 0.04$ ev (curve *I*) the value of τ approaches $\sim 5\times10^4\ \tau_{p0}$, for $\mathcal{E}_u - \mathcal{E}_t = 0.22$ ev $\tau_{\max} \sim 10^2\ \tau_{p0}$, and for $\mathcal{E}_t = 0$ τ does not exceed $2\,\tau_{p0}$.

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Bipolar Diffusion of Current Carriers in the Presence of Deep Traps

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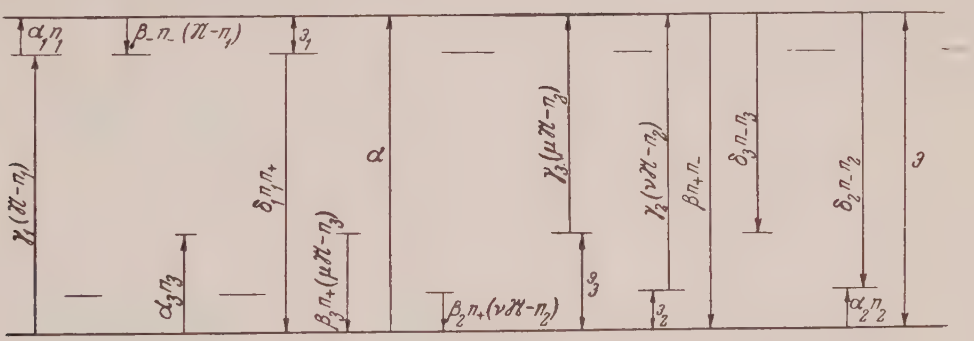
We have investigated bipolar diffusion in the presence of traps, taking into consideration the dependence of minority carrier lifetime on the population of the traps, and have established criteria for the validity of the linear recombination law. An equation was derived for the concentration distribution of minority carriers which obey a nonlinear recombination law

IN papers on the bipolar diffusion of current carriers in semiconductors it is usually assumed that when the concentration of minority carriers p is considerably lower than the equilibrium concentration of majority carriers n_0, a linear recombination law with a definite lifetime τ holds true. However, the linear recombination law can possibly be violated even when $p << n_0$. This will occur in the presence of hole-trapping levels which are widely separated from the valence band (for definiteness the semiconductor will hereafter be assumed to be n-type), in which case the holes will have a long lifetime on these levels before recombination. Thus the concentration of trapped holes and consequently, the number of recombinations with electrons will no longer be a linear function of the number of free holes. Lashkarev[1] has found a criterion for the validity of a linear recombination law in this case. However, he did not in the same paper consider bipolar diffusion in the case of nonlinear recombination.



Semiconductor energy scheme: n_1 = concentration of electrons on donors; n_2 and n_3 = concentrations of holes on acceptors and in traps. Arrows indicate the directions of electron transitions. Transition probabilities are indicated alongside the arrows.

Additional holes can be created either by irradiation of the semiconductor or by injection through a $p-n$ junction. Our further discussion will concern radiation-induced "photoholes" without limiting its generality.

The figure shows schematically the location of the usual donor levels *1*, the acceptor levels *2* and the deep traps *3*. Inscriptions alongside the arrows show the number of corresponding transitions per unit time. In Ref. 2 an expression has been derived for the divergence of the electronic current, ej_-, given by the difference between the numbers of excitations and recombinations of conduction electrons, when only acceptor and donor levels are present. A more general expression can be obtained similarly when deep traps are present. Just as in Ref. 2, where it is assumed that there is a relatively rapid exchange between donor levels and the (electron conduction band, and between acceptor levels and the valence band, it can be assumed that the recombination of captured holes with conduction electrons goes through a "bottleneck", but that capture and liberation of holes take place much more rapidly. Then instead of Eq. (8) of Ref. 2, we obtain

$$\operatorname{div} j_- = -\mathfrak{N}^2\left[\beta + \frac{\delta_1}{N+a_1} + \frac{\nu\delta_2}{z+a_2} + \frac{\mu\delta_3}{z+a_3}\right](Nz-b), \tag{1}$$

where $\mathfrak{N}$, $\nu\mathfrak{N}$ and $\mu\mathfrak{N}$ are the concentrations of donors, acceptors and deep traps; and $N\mathfrak{N}$ and $z\mathfrak{N}$ are the concentrations of free electrons and traps. The meaning of the recombination coefficients $\beta, \delta_1, \delta_2$ and δ_3 is clear from the Figure. In addition, we have used the notation

$$a_1 = (B_-/\mathfrak{N})\, e^{-\vartheta_1/kT}; \quad a_{2,3} = (B_+/\mathfrak{N})\, e^{-\vartheta_{2,3}/kT};$$
$$b = (B_+B_-/\mathfrak{N}^2)\, e^{-\vartheta/kT}; \quad B_\pm = 2h^{-3}(2\pi m_\pm kT)^{3/2}, \tag{2}$$

assuming that $\mathfrak{N} << B\pm$ and that ϑ_1 and ϑ_2 are not too large compared with kT, we neglect z and N compared with a_3. When to (1) are added the equations which define the electron current j_- and the hole current j_+ as well as Poisson's equation we obtain a complete set of equations; for the one-dimensional case and with dimensionless quantities, these are

$$dN/d\xi = Ny + \varphi, \tag{3a}$$

$$dz/d\xi = -zy - (\lambda - \varphi)/K, \tag{3b}$$

$$dy/d\xi = N - z - 1 + (\nu) - \mu z/(z+a_3) + (\mu), \tag{3c}$$

$$\frac{d\varphi}{d\xi} = \left[A + \frac{D}{z+a_3}\right](Nz-b), \tag{3d}$$

where

$$\varphi = -j_-/\mathfrak{N}\vartheta_-\varkappa; \quad \lambda = I/e\mathfrak{N}\vartheta_-\varkappa; \quad y = (e/\varkappa kT)\, dV/dx,$$
$$\varkappa = \sqrt{\frac{4\pi e^2\mathfrak{N}}{\varepsilon kT}}; \tag{4}$$
$$A = \frac{\varepsilon}{4\pi e u_-}\left[\beta + \frac{\delta_1}{a_1} + \frac{\delta_2}{a_2}\right]; \quad D = \frac{\varepsilon\mu\delta_3}{4\pi e u_-}.$$

Here I is the current density, ϑ_- and u_- are the diffusion coefficient and mobility for electrons, V is the electric potential, $\xi = \varkappa x$ is a dimensionless coordinate, ϵ is the dielectric constant and $K = u_+/u_-$. The quantities ν and μ in brackets may be omitted when acceptors and hole traps containing electrons are neutral (we consider ionized donors to be positively charged).

When we confine ourselves to illumination under which the concentration of photocurrent carriers is considerably smaller than the concentration of dark carriers, the case under consideration of possible saturation of traps by photoholes can be realized in practice, clearly, only in an impurity semiconductor ($z_{\text{dark}} << N_{\text{dark}}$). Therefore we shall assume that everywhere $b << 1$ and $z << 1$.

In accordance with the usual experimental arrangement, when there is no current through the specimen (subject to a photoelectromotive force) $\lambda = 0$, while φ and z can be considered small first order quantities. Since in a large part of its volume the specimen can be considered almost exactly neutral, it follows from Eq. (3c) that

$$N = 1 + z - (\nu) - (\mu) + \mu z/(z+a_3). \tag{5}$$

Eliminating N and $dz/d\xi$ from (3a), (3b) and (5) we obtain the field y which we substitute in (2b):

$$\frac{dz}{d\xi} = \left[1 - \frac{\mu a_3 z(z+a_3)^{-2} + (1-K)z}{1 + 2z - (\nu) - (\mu) + \mu z(z+a_3)^{-1} + \mu a_3 z(z+a_3)^{-2}}\right] \times \frac{\varphi}{K}. \tag{6}$$

Since $za_3/(z+a_3)^2 \leq 1/4$, when $\mu < 1 - (\mu) - (\nu)$, the fraction within the square brackets can be omitted as an approximation by comparison with $1 - (\nu) - (\mu)$.
The above inequality denotes that the trap concentration is smaller than the conduction electron for $z=0$, as can be seen from (5). Following the indicated simplification we obtain instead of (6)

$$dz/d\xi = \varphi/K. \tag{7}$$

Eliminating the electronic current φ from (7) and (3d), we have

$$\frac{d^2z}{d\xi^2} = \frac{1}{K}\left(A + \frac{D}{z+a_3}\right) \times \left[z\left(1 - (\nu) - (\mu) + z + \frac{\mu z}{z+a_3}\right) - b\right]. \tag{8}$$

We determine the dark concentration of holes $\bar{z}$ from the condition that the right-hand side of Eq. (8) vanishes and we set $z = \bar{z} + \zeta$. For ζ we obtain the equation

$$\frac{d^2\zeta}{d\xi^2} = \frac{1}{K}\left(A + \frac{D}{\bar{z} + a_3 + \zeta}\right)\left(1 - (\nu) - (\mu) + 2\bar{z} + \zeta + \frac{\mu \bar{z} a_3}{(\bar{z} + a_3)(\bar{z} + a_3 + \zeta)} + \frac{\mu(\bar{z} + \zeta)}{\bar{z} + a_3 + \zeta}\right)\zeta, \quad (9)$$

where $\bar{z}$ is defined by the relation

$$\bar{z}\left(1 - (\nu) - (\mu) + \bar{z} + \frac{\mu\bar{z}}{\bar{z} + a_3}\right) = b. \quad (10)$$

The next-to-last term in the second bracket of (9) can be omitted by comparison with $1 - (\nu) - (\mu)$, just as was done in Eq. (6). Following such simplification the equation can be reduced to quadrature. With the boundary conditions $\zeta = \zeta = 0$ for $\xi = \infty$ and $\varphi = \varphi_0$ for $\xi = 0$, the solution is

$$\xi = -\int_{\zeta_0}^{\zeta} d\zeta / \sqrt{F(\zeta)}, \quad (11)$$

$$F(\zeta) = \beta^2\zeta^2 + 2\beta^2\gamma\zeta + 2\left[\beta^2\gamma a_3' + \frac{\mu D}{K}(\bar{z} - 3a_3')\right]\ln\left(\frac{a_3' + \zeta}{a_3'}\right) + 2\frac{\mu D}{K}\frac{(a_3' - \bar{z})}{a_3' + \zeta}\zeta + \frac{2A}{3K}\zeta^3. \quad (12)$$

Here the following notation has been introduced:

$$a_3' = a_3 + \bar{z};\ \beta = [(A/K)(1 - (\nu) - (\mu) + \mu + 2\bar{z}) + D/K]^{1/2};$$
$$\gamma = (1/K\beta^2)[A\mu(\bar{z} - a_3') + D(1 - (\nu) - (\mu) + \mu - a_3' + 2\bar{z})]. \quad (13)$$

The constant of integration ζ_0 is determined from Eqs. (7) and (11) for the point $\xi = 0$ in terms of the hole current density $-\varphi_0$ which enters an unilluminated semiconducting region from the illuminated region $\xi < 0$. In particular, when illumination is produced by a narrow luminous probe whose width is considerably smaller than the diffusion length, $-\varphi_0$ is eqal to half the number of holes created by light in unit time in a unit area which is perpendicular to the ξ axis. This condition is expressed by

$$\sqrt{F(\zeta_0)} = -\varphi_0/K. \quad (14)$$

The integral in (11) can be calculated in the two limiting cases.

1. The case of small photohole densities ($\zeta << a'_3$). Then, expanding $F(\zeta)$ in powers of ζ / a'_3 and stopping at the quadratic terms, we obtain after integration

$$\zeta = Ce^{-\alpha\xi},\ \alpha = \left[\frac{1}{K}\left(A + \frac{D}{a_3'}\right)\left(1 - (\nu) - (\mu) + 2\bar{z} + \frac{\mu\bar{z}}{a_3'}\right)\right]^{1/2}; \quad (15)$$

$1/a$ is the dimensionless diffusion length.

2. The case of high photohole densities ($\zeta >> a'_3$). In this case only the first two terms of Eq. (12) can be retained. Integration then gives

$$\zeta = \gamma\left[\left(\frac{\zeta_0}{\gamma} + 1\right)\operatorname{ch}\beta\xi - \sqrt{\frac{\zeta_0^2}{\gamma^2} + 2\frac{\zeta_0}{\gamma}}\operatorname{sh}\beta\xi - 1\right]. \quad (16)$$

According to (14)

$$\beta\sqrt{\zeta_0^2 + 2\gamma\zeta_0} = -\varphi_0/K. \quad (17)$$

When $\zeta_0 >> a'_3$ it is possible to use (16) approximately in the region $\zeta > a'_3$ and (15) for $\zeta < a'_3$, and to determine the constant C in (15) from the continuity of ζ at the point $\zeta = a'_3$, where both criteria are in agreement. When $\zeta_0 << a'_3$ Eq. (15) is valid everywhere and

$$C = \zeta_0 = -\varphi_0/K\alpha. \quad (18)$$

As can be seen from (16), ζ is a diminishing function of ξ as far as the point $\xi_{\lim}$ which is defined by the equation

$$\operatorname{ch}\beta\xi_{\lim} = (\zeta_0/\gamma) + 1; \quad (19)$$

at this point ζ and $d\zeta / d\xi$ vanish simultaneously. The solution has physical meaning only in this case and can be connected continuously with the solution in (15).

From a comparison of (15) and (16) it can be seen that for $\zeta < a'_3$ the concentration decreases exponentially with the exponent $\alpha\xi$, while for $\zeta < a'_3$ it decreases more slowly whan an exponential with the exponent $\beta\xi$. At the same time we have from the definitions of α and β: /

$$\alpha^2 = \beta^2 + \frac{D}{Ka'_3}\left[1 - (\nu) - (\mu) + 2\bar{z} + \frac{\mu\bar{z}}{a'_3} - a'_3\right] - \frac{\mu A}{K}\frac{a'_3 - \bar{z}}{a'_3}. \tag{20}$$

When a'_3 is small α can be very much larger than β. Thus, when $a'_3 << 1$, in the region $a'_3 < \zeta << 1$, which is usually considered linear, the linear recombination law can be violated seriously and the mean hole lifetime can increase. This must evidently occur frequently in semiconductors. Thus, for example, for $\mathfrak{N} = 10^{16}$ cm^{-3}, $\text{Э}_3 = 0.3$ ev and at room temperature, $a_3 = 0.014\,(m_+/m_0)^{3/2}$.

Consequently, in investigations of photoconduction and the photoelectromotive force even in the "linear region", where the concentration of light-induced carriers comprises only a small percentage of the concentration of dark carriers, the condition $\zeta << a'_3$ can be violated, at least near the illuminated region. To fulfill the linear recombination law throughout the crystal it follows from (15) and (18) that the following condition must be satisfied:

$$\frac{J\gamma}{2\hbar\omega}\frac{l}{d}\frac{L_+}{\vartheta_+} \ll B_+\left(e^{-\text{Э}_3/kT} + \frac{B_-}{\mathfrak{N}}e^{-\text{Э}/kT}\right); \tag{21}$$

where J and ω are the intensity and frequency of the incident light, γ is the quantum yield of the photoconductive effect, l is the width of the illuminated region along the x-axis and d is the thickness of the specimen.

When the trapping levels *3* are located close to the middle of the forbidden zone and the probability of hole capture is of the same order as the probability of recombination between a captured hole and an electron, instead of the last term in the square bracket of Eq. (1), a calculation gives

$$\frac{\mu\delta_3}{(b_3 + N)(\delta_3/\beta_3) + a_3 + z}, \tag{22}$$

where

$$b_3 = (B_-/\mathfrak{N})\,e^{-(\text{Э}-\text{Э}_3)/kT}. \tag{23}$$

Since $N \sim 1$, for $\delta_3 \sim \beta_3$ it is always possible to neglect z in the denominator within the linear region ($z << 1$), and there will be no effects associated with trap saturation.

The authors consider it their pleasant duty to acknowledge their indebtedness to E. G. Miseliuk, who made them aware of this problem.

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A Quasi-Unidimensional Interpretation of the Hydrodynamic Theory of Multiple Particle Production

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The hydrodynamic theory of multiple particle production developed by L. D. Landau is based on introducing two stages of liquid breakup——unidimensional motion and conical breakup. The validity range of the second stage is exceedingly difficult to estimate. This work investigates a variant of hydrodynamic theory, in which only the unidimensional stage is considered. This variant is shown to give very good approximation for final temperatures $T_k \sim 1.5\,\mu - 2\mu$. At $T_k \sim \mu$ the unidimensional approximation (particularly for slow secondary particles) gives a result that is merely of the right order of magnitude. The dependence of the fastest particle on T_k is also investigated. It turns out that the condition $T_k \lesssim \mu$ must be satisfied for the calculated velocity to agree with the experimentally-observed one. This leads to the preliminary deduction that when $T_k \sim \mu$ the interaction cross section of the secondary particles (apparently pi-mesons) is of the same order of magnitude as the geometric cross section.

1. Landau,[1] on the basis of an idea expressed by Fermi,[2] developed a hydrodynamic theory for the interaction of high-energy particles. The characteristic parameter of the theory is the value of the final temperature T_k, determined by the cross section σ for the interaction of the secondary particles. Landau assumes $T_k \sim \mu$* (μ is the meson mass). However, this choice is by no means unique; if the cross section σ diminishes with diminishing energy, T_k may become much greater than μ.[3] On the other hand, if the cross section is $\sigma \sim 1/\mu$, then $T_k \sim \mu$.[4] The value of σ is thus closely linked to the value of the final temperature. To date there are no direct experimental data concerning the character of the meson–meson interaction. One must therefore choose a different path, namely: first obtain the value of T_k by comparing the results of the hydrodynamic theory with the experimental data, and then deduce the magnitude of σ.

We shall consider in this investigation the variation of the energy characteristics of the elementary act with T_k. We encounter here the very difficult problem of accounting for the lateral breakup of the liquid. The conical breakup introduced by Landau for this purpose is a rather rough approximation, the validity of which is moreover difficult to estimate. We shall therefore consider another approximation that is more advantageous for our purposes; in the calculation of the energy characteristics (more accurately, the four-velocity u), we neglect the lateral breakup entirely (quasi-unidimensional approximation).

2. For our purpose it is quite important to investigate the portion of nuclear matter carrying the principal energy fraction (the so-called leading edge). However, it is precisely to this section that the Landau results are inapplicable, for the condition $\xi >> \Delta$* on which his solution is based is not satisfied here. We therefore first obtain a solution for the forward front. The remaining computations are based on the concept that if a very high energy is concentrated in the volume in which the process occurs, the transverse components of the four-velocity u are much smaller than the longitudinal one, and the energy of the particles is therefore determined almost completely by the unidimensional motion. Inasmuch as Khalatnikov[5] (cf. also Ref. 6) obtained an accurate solution for the leading edge, our results are approximate only in that the transverse components of u have been neglected.

It is thus necessary to determine first of all the applicability limits of the unidimensional solution. Landau's estimate of this limit is expressed in the form of the inequality $t\,\xi << a^2$. As was already noted, he employed the unidimensional solution; we shall refine this estimate, relying on Khalatnikov's solution of the unidimensional problem. The equations of motion of an ultra-relativistic ideal liquid are given by the following equations:

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*We employ here a system of units in which $h=c=M=1$ (M is the nucleon mass). The temperature is expressed in energy units.

*Hereinafter we shall employ Landau's notation.

$$\partial T_{ik}/\partial x_k = 0, \tag{1}$$

$$T_{ik} = (\varepsilon/3)(4u_i u_k + g_{ik}), \tag{2}$$

where ϵ is the energy density in the intrinsic coordinate system, u_i the four-velocity components, $g_{11} = g_{22} = g_{33} = 1$, $g_{00} = -1$, and $g_{ik} = 0$ if $i = k$; i and k run through the values 0, 1,2 and 3.* The index 0 denotes the temporal components, and the index 1 denotes the components along the motion of the primary particles (axis of motion). In the case of unidimensional motions, the indices i and k are either 0 or 1.

Let us write the solution given by Khalatnikov for this case in its parametric form

$$x_0 = t = e^{-y}\left(\frac{\partial\chi}{\partial y}\cosh\rho - \frac{\partial\chi}{\partial\rho}\sinh\rho\right), \tag{3}$$

$$t - x_1 = \xi = e^{-y}\left(\frac{\partial\chi}{\partial y} + \frac{\partial\chi}{\partial\rho}\right)(\cosh\rho - \sinh\rho), \tag{4}$$

$$\chi = \sqrt{3}\,\Delta e^{y}\int\limits_{-\rho/\sqrt{3}}^{y} e^{-2z} I_0\left(\sqrt{z^2 - \rho^2/3}\right) dz, \tag{5}$$

where $y = \ln(T/T_0)$, $u_0 = \cosh\rho$, T_0 the initial temperature, T the running temperature of the liquid element, and I_0 Bessel's function for imaginary argument. It will be more convenient for what follows to write the derivatives $\partial\chi/\partial y$ and $\partial\chi/\partial\rho$ in explicit form:

$$\frac{\partial\chi}{\partial y} = \sqrt{3}\,\Delta e^{-y} I_0\left(\sqrt{y^2 - \rho^2/3}\right) \tag{6}$$

$$- e^{y}\int\limits_{\rho/\sqrt{3}}^{-y} e^{2z} I_0\left(\sqrt{z^2 - \rho^2/3}\, dz,\right.$$

$$\frac{\partial\chi}{\partial\rho} = e^{y}\Delta\left[e^{2\rho/\sqrt{3}}\right.$$

$$+ \frac{\rho}{\sqrt{3}}\int\limits_{\rho/\sqrt{3}}^{-y} e^{2z} I_1\left(\sqrt{z^2 - \rho^2/3}\right)\frac{dz}{\sqrt{z^2 - \rho^2/3}}\,.$$

Consider next, two cases: a) $y^2 - \rho^2/3 \lesssim 1$ (leading edge) and b) $y^2 - \rho^2/3 >> 1$, corresponding to the slowest particles. As will be seen below, both cases overlap in practice. Expanding the I functions and restricting ourselves to the first three terms of this series, we obtain for case a):

$$t = u\Delta\left\{e^{-2y}\left[\sqrt{3} + \frac{\sqrt{3}}{4}\left(y^2 - \frac{\rho^2}{3}\right)\right.\right. \tag{7}$$

$$- \frac{6+\rho}{4\sqrt{3}} + \frac{\sqrt{3}}{64}\left(y^2 - \frac{\rho^2}{3}\right)^2$$

$$\left. - \frac{12+\rho}{16\sqrt{3}}\left(\frac{y^2}{2} + \frac{y}{2} + \frac{1}{4} - \frac{\rho^2}{6}\right)\right]$$

$$\left. - e^{2\rho/\sqrt{3}}\left[1 - \frac{6+\rho}{4\sqrt{3}} + \frac{12+\rho}{16\sqrt{3}}\left(\frac{\rho}{2\sqrt{3}} - \frac{1}{4}\right)\right]\right\},$$

$$\xi = \frac{\Delta}{2u}\left\{e^{-2y}\left[\sqrt{3} + \frac{\sqrt{3}}{4}\left(y^2 - \frac{\rho^2}{3}\right)\right.\right. \tag{8}$$

$$- \frac{6-\rho}{4\sqrt{3}} + \frac{\sqrt{3}}{64}\left(y^2 - \frac{\rho^2}{3}\right)$$

$$\left. - \frac{12-\rho}{16\sqrt{3}}\left(\frac{y^2}{2} + \frac{y}{2} + \frac{1}{4} - \frac{\rho^2}{6}\right)\right]$$

$$\left. + e^{2\rho/\sqrt{3}}\left[1 + \frac{6-\rho}{4\sqrt{3}} - \frac{12-\rho}{16\sqrt{3}}\left(\frac{\rho}{2\sqrt{3}} - \frac{1}{4}\right)\right]\right\}.$$

The relationship $\sinh\rho \sim \cosh\rho \sim e^{\rho}/2$ was used in the derivation of (7) and (8). If condition a) is satisfied, (7) and (8) converge very rapidly. Furthermore, it can be shown that the denominators of the series terms increase so rapidly that the contribution of the discarded terms of power $k \geq r$ is only on the order of the rth term.

Particularly simple are the equations obtained for the limits of the self-modeling and nontrivial motion ($\alpha = -y - \rho/\sqrt{3} = 0$). In this case

$$t = \frac{\sqrt{3}-1}{2}\Delta\exp\left\{\frac{\rho}{\sqrt{3}}(2+\sqrt{3})\right\}; \tag{9}$$

$$\xi = (\sqrt{3}+1)\Delta\exp\left\{\frac{\rho}{\sqrt{3}}(2-\sqrt{3})\right\},$$

$$u^2 = [(\sqrt{3}+1)/2(\sqrt{3}-1)]\,t/\xi; \tag{10}$$

$$\varepsilon = \varepsilon_0\Delta^2/t\xi;$$

$$t = \left(\frac{\sqrt{3}-1}{2}\right)(\sqrt{3}+1)^{-(2+\sqrt{3})/(2-\sqrt{3})}$$

$$\times\ \xi^{(\sqrt{3}+2)(2-\sqrt{3})}\Delta^{-2\sqrt{3}/(2-\sqrt{3})}.$$

If $\alpha << 1$, then, restricting ourselves to the first terms of the series (7) and (8) we can obtain a more general expression, valid not merely at $\alpha = 0$:

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*The equation of state $\epsilon/3 = p$ is used in the derivation of (2).

$$u^2 = {}^1/_4\,(t\xi)^{\sqrt{3}/2}\,\Delta^{-\sqrt{3}}, \qquad (11)$$

$$\varepsilon \doteq \left(\frac{2}{\sqrt{3}-1}\right)^{2(2-\sqrt{3})/\sqrt{3}}$$

$$\times\,(\sqrt{3}+1)^{2(2+\sqrt{3})/\sqrt{3}}\,\varepsilon_0\Delta^6\xi^{-(3\sqrt{3}+4)/\sqrt{3}}$$

$$\times\ t^{-(3\sqrt{3}-4)/\sqrt{3}}.$$

Let us consider further the case b):

$$y^2 - \rho^2/3. \qquad (12)$$

For this purpose we employ the asymptotic expansion of the Bessel function:

$$I_p(\omega) \sim \frac{e^{\omega}}{\sqrt{2\pi\omega}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2\omega)^k}\,\frac{\Gamma(p+k+{}^1/_2)}{k!\,\Gamma(p-k+{}^1/_2)}. \qquad (13)$$

We neglected the term e^{ω} in equation (13); this is quite appropriate at $\omega > 1$.

Before inserting the series (13) into (6), let us evaluate the integral

$$J_p = \int_{\rho/\sqrt{3}}^{-y} e^{2z}\,\frac{I_p(\sqrt{z^2-\rho^2/3})}{(\sqrt{z^2-\rho^2/3})^m}\,dz.$$

Let us break this integral up into two. In the first (J_{p1}), for which $z^2 - \rho^2/3 > 1$ (corresponding to $\alpha \gtrsim 1/\rho$), we shall employ the asymptotic expansion; in the second (J_{p2}; $z^2 - \rho^2/3 < 1$) we shall employ the series (6). Since the contribution of the second integral is slight, it can be adequately represented by the first terms of series (6);

$$J_{11} \sim 2J_{12} \sim \frac{1}{2}\,e^{2\rho/\sqrt{3}}\,(e^{\alpha}-1) \sim \frac{1}{\rho}\,e^{2\rho/\sqrt{3}} \qquad (14)$$

The integral J_{12} reduces to the sum of integrals

$$R_k \sim \int \frac{\exp\{2z + \sqrt{z^2-\rho^2/3}}{(z^2-\rho^2/3)^{{}^1/_2\,(k+m+{}^1/_2)}}\,dz.$$

The important factor in this integral is the exponential term. Let us therefore first evaluate the integral

$$\Phi(z) = \int \exp\{2z + \sqrt{z^2-\rho^2/3}\}\,dz,$$

and then make allowances for the effect of the denominator. We seek $\Phi(z)$ in the form

$$\Phi(z) = \left[\frac{1}{2 + z/\sqrt{z^2-\rho^2/3}} + \varphi(z)\right] \qquad (15)$$

$$\times\ \exp\{2z + \sqrt{z^2-\rho^2/3}\},$$

where $\varphi(z)$ satisfies the equation

$$[2 + z/\sqrt{z^2-\rho^2/3}]\,\varphi(z) + \varphi'(z) \qquad (16)$$

$$+\ \rho^2/3\sqrt{z^2-\rho^2/3}\,(2\sqrt{z^2-\rho^2/3} + z)^2 = 0.$$

Assuming $z^2 - \rho^2/3 >> 1$, we can approximate (16) by

$$3\varphi(z) + \varphi'(z) + \rho^2/27z^3 = 0, \qquad (17)$$

hence

$$\varphi(z) = \frac{\rho^2}{27}\,e^{-3z}\int \frac{e^{3z}}{z^3}\,dz. \qquad (18)$$

Using the asymptotic expansion

$$\int \frac{e^{rz}}{z^k}\,dz \sim \frac{e^{rz}}{rz^k}\left[1 + \frac{k}{2z} + \frac{k(k+1)}{(rz)^2}\right. \qquad (19)$$

$$\left. +\ \ldots + \frac{k(k+1)\ldots(k+i)}{(rz)^{i+1}} + \ldots\right]$$

and restricting ourselves to the first term only, which is quite appropriate as long as we are interested in relatively small k and large z, we can finally write

$$\varphi(z) \sim \rho^2/81z^3. \qquad (20)$$

The function $\varphi(z) << \rho^2/54\,z^2$ and can therefore be neglected in (15); in our approximation

$$\Phi(z) \sim {}^1/_3\,(1 - \rho^2/18z^2) \qquad (21)$$

$$\times\ \exp\{2z + \sqrt{z^2-\rho^2/3}\}.$$

If $z^2 - \rho^2/3 \sim 1$, we cannot represent (16) by (17); analysis of (16) shows that in this case $\varphi(z) \sim 1/(2+z)$ and neglecting this function results therefore in an error by a factor of approximately 2. In the second (logarithmic) approximation, the factor in front of the exponent in (15) must be set equal to unity.

Let us evaluate the role of the denominator in the principal integral. Using an asymptotic expansion analogous to (19) and restricting ourselves to the first term, we can obtain

$$R_k \sim {}^1/_3\,(1 - \rho^2/18z^2)\,(z^2 - \rho^2/3)^{-{}^1/_2(k+m+{}^1/_2)} \qquad (22)$$

$$\times\ \exp\{3z - \rho^2/6z^2\};$$

at $z^2 - \rho^2/3 >> 1$

$$R_k \sim {}^1/_3\left(1 + \frac{m+k+{}^1/_6}{6z^2}\right) z^{-(k+m)-{}^1/_8} \qquad (23)$$

$$\times\ \exp\{3z - \rho^2/6z^2\};$$

if $z^2 - \rho^2/3 \sim 1$

$$R_k \sim (2+z)^{-1}(z^2-\rho/3)^{-1/2(k+m\pm 1/2)} \tag{24}$$
$$\times \exp\{2z + \sqrt{z^2-\rho^2/3}\}.$$

We shall assume hereinafter that (23) represents the upper limit of the integral, and (24) the lower limit. This is equivalent to the condition $y^2 - \rho^2/3 >> 1$. Let us next evaluate

$$J_{11} - R_0(\sqrt{(\rho^2/3)+1}) \sim \frac{e^{2\rho/\sqrt{3}}}{\rho(2+z)}(2+z-\rho).$$

Since $z \sim \rho/\sqrt{3}$ in this case, and we are interested in $z \sim 2 — 4$ then

$$J_{11} - R_0\left(\sqrt{\frac{\rho^2}{3}+1}\right) \sim \sqrt{3}e^{2\rho/\sqrt{3}}/\rho(2\sqrt{3}+\rho).$$

It follows from this that

$$J_{11} - R_0(\sqrt{(\rho^2/3)+1})/R_0(-y) \sim e^{-z}.$$

Thus, restricting ourselves to an approximate accuracy of 10 – 15 %, we can neglect R_0 and the integrals J_{11} and J_{12}.

Examination of (13) shows that to obtain the same accuracy, approximately 10%, we need use only the first term of the series.

Let us now derive several specific equations. Neglecting the factors in front of the exponents in (23) and (24) and the quantities J_1, $R\,[\sqrt{(\rho^2/3)}+1]$, and $\partial X/\partial\rho$, we get

$$\tau \sim \rho - 2y + \sqrt{y^2-\rho^2/3}, \tag{25}$$
$$\eta \sim -\rho - 2y + \sqrt{y^2-\rho^2/3},$$

where

$$\tau = \ln(t/\Delta), \quad \eta = \ln(\xi/\Delta). \tag{26}$$

As can be seen from the preceding estimates, the errors due to the items neglected are quite large if $y^2 \sim \rho^2/3$; the relationships (25) and (26) are merely of the right order of magnitude

Solving both (25) and (26) for ρ and y and then using $u = e^{\rho}/2$ and $\epsilon = \epsilon_0 e^{4y}$, we obtain

$$u^2 = t/\xi, \tag{27}$$
$$\varepsilon = \varepsilon_0 \exp\{-4/3(n+\tau-\sqrt{\eta\tau})\}.$$

Equations (27) were first introduced by Landau[1].* Somewhat simpler expressions can be obtained if $y^2 >> \rho^2/3$. In this case, neglecting the factor $e^{\rho\sqrt{3}y}$, we can write

$$e^{\tau} = \sqrt{3/8\pi}e^{-3y}e^{\rho}y^{-1/4}, \tag{28}$$
$$e^{n} = \sqrt{3/2\pi}e^{-3y}e^{-\rho}y^{-1/4}.$$

Putting $y^{-1/2} = 1$, which usually introduces an error of the order of 2, we have

$$\varepsilon \approx (\varepsilon_0/3)(\Delta^2/t\xi)^{2/3}; \quad u^2 \sim t/\xi. \tag{29}$$

3. Let us evaluate the applicability limits of the unidimensional solution. For this purpose let us write Eq. (1) as :

$$4\frac{\partial}{\partial t}(\varepsilon u^2) + \frac{\partial\varepsilon}{\partial\xi} + 8\frac{\partial}{\partial x_2}(\varepsilon u^2\vartheta) = 0; \tag{30}$$

$$\frac{\partial\varepsilon}{\partial t} + \frac{\partial}{\partial\xi}\left(\frac{\varepsilon}{u^2}\right) + 4\frac{\partial}{\partial x^2}(\varepsilon\vartheta) = 0; \tag{31}$$

$$4\frac{\partial}{\partial t}(\varepsilon u^2\vartheta) + 4\frac{\partial}{\partial x_1}(\varepsilon u^2\vartheta) + \frac{\partial\varepsilon}{\partial x_2} = 0. \tag{32}$$

The following was assumed in the derivation of these equations: $u_0 \sim u_1 = u$; $u_2 = u_3 = u\,\vartheta$; $\partial/\partial x_2 = \partial/\partial x_3$ *. In addition, the term discarded in (32) is considerably smaller than the term remaining. Under these assumptions, the fourth equation of (1) becomes identical wthh (32).

Equations (31) and (32) differ from the equations for the unidimensional problem in their last terms. We shall assume that in (32) these terms are small compared with the remaining ones. We shall therefore write equation (32), which relates small components of the tensor T, in an approximate form, depending on what section of the liquid is under investigation. Let us consider first the leading edge, for which $x_1 \sim t$. Since $\epsilon(t=0) = \epsilon(x_1=0) = \epsilon_0$ and $u(t=0) = u(x_1=0) \sim 1$, then $\partial/\partial t \sim \partial/\partial x_1$ and Eq. (32) can be rewritten as:

$$8(\partial/\partial t)(\varepsilon u^2\vartheta) + \partial\varepsilon/\partial x_2 = 0. \tag{32a}$$

We seek a solution in the form

$$u^2 = u_1^2(t,\xi) + \varphi_1(t,\xi);$$
$$\varepsilon = e^{-x_2/a}[\varepsilon_1(t,\xi) + \varphi_2(t,\xi)]$$

(ϵ_1 and u_1 satisfy the equations describing the unidimensional motion), corresponding to an initial exponential radial distribution of the energy density. Assuming $\varphi_1 \ll u_1^2$, and $\varphi_2 \ll \epsilon_1$ and making use of the relative smallness of the last terms of (30) and (31), we obtain the following system:

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*The Landau solution is derived from the asymptotic expansion in Ref. 5.

*The equalities $u_2 = u_3$ and $\partial/\partial x_2 = \partial/\partial x_3$ follow from the axial symmetry.

$$4\frac{\partial}{\partial t}(\varphi_1\varepsilon_1+\varphi_2 u_1^2)+\frac{\partial\varphi_2}{\partial\xi}-\frac{1}{a^2}\int\varepsilon_1 dt=0, \tag{33a}$$

$$\frac{\partial\varphi_2}{\partial t}+\frac{\partial}{\partial\xi}\left[\frac{\varphi_2}{u_1^2}-\frac{\varepsilon_1\varphi_1}{u_1^4}\right]-\frac{1}{2a^2u^2}\int\varepsilon_1 dt=0, \tag{33b}$$

$$\vartheta=\frac{1}{8a\varepsilon_1 u_1^2}\int\varepsilon_1 dt. \tag{33c}$$

We next seek a solution in the form

$$\varphi_1=(C_1\xi t/a^2)\,u_1^2,\quad \varphi_2=(C_2\xi t/a^2)\,\varepsilon_1.$$

Such a solution satisfies the initial conditions $\varphi(t=0)=\varphi(\xi=0)=0$. Using (14) we get

$$\frac{11-4\sqrt{3}}{2\sqrt{3}} \tag{34a}$$

$$\times(C_1+C_2)\,t^{(11-6\sqrt{3})/2\sqrt{3}}\,\xi^{-(4\sqrt{3}+5)/2\sqrt{3}}\,\Delta^{-\sqrt{3}}$$

$$-\frac{2\sqrt{3}+4}{\sqrt{3}}C_2 t^{2(2-\sqrt{3})/\sqrt{3}}$$

$$\times\xi^{-(3\sqrt{3}+4)/\sqrt{3}}$$

$$-\frac{\sqrt{3}}{2(2-\sqrt{3})}t^{2(2-\sqrt{3})/\sqrt{3}}\,\xi^{-(3\sqrt{3}+4)/\sqrt{3}}=0,$$

$$2(2-\sqrt{3})\,C_2 t^{(4-3\sqrt{3})/\sqrt{3}}\,\xi^{-2(2+\sqrt{3})/\sqrt{3}} \tag{34b}$$

$$-(11+4\sqrt{3})(C_2-C_1)\,t^{(5-4\sqrt{3})/2\sqrt{3}}$$

$$\times\xi^{-(11+6\sqrt{3})/2\sqrt{3}}\,\Delta^{\sqrt{3}}$$

$$-\frac{3}{2-\sqrt{3}}t^{(5-4\sqrt{3})/2\sqrt{3}}\,\xi^{-(11+6\sqrt{3})/2\sqrt{3}}\,\Delta^{\sqrt{3}}=0.$$

Equations (34) cannot be solved for $C=$ constant. However, we can assume that the last equation in (10) is approximately satisfied in the vicinity of the leading edge. Then, expressing $t^{(2-\sqrt{3})/2}$ [in the first terms of (34a) and (34b)]* in terms of ξ, in accordance with (10), we obtain a simple equation from which it follows that

$$\varphi_1\sim(\xi t/3a^2)\,u_1^2;\quad \varphi_2\sim(\xi t/4a^2)\,\varepsilon_1. \tag{35}$$

We next consider a section in which the condition $\rho^2/3 \ll y^2$ is satisfied. In this case it is convenient to write Eq. (32) as :

$$4\frac{\partial}{\partial t}(\varepsilon u^2\vartheta)+2\frac{\partial}{\partial\xi}(\varepsilon\vartheta)+\frac{\partial\varepsilon}{\partial x_2}=0. \tag{32b}$$

In the region under investigation $t\sim\xi$ and $\partial/\partial t \sim \partial/\partial\xi$, and therefore we have in lieu of (33)

$$4\frac{\partial}{\partial t}(\varphi_1\varepsilon_1+\varphi_2 u_1^2)+\frac{\partial\varphi_2}{\partial\xi}-\frac{4}{9a^2}\int\varepsilon_1 dt=0; \tag{36a}$$

$$\frac{\partial\varphi_2}{\partial t}+\frac{\partial}{\partial\xi}\left[\frac{\varphi_2}{u_1^2}-\frac{\varepsilon_1\varphi_1}{u_1^4}\right]-\frac{2}{9a^2u_1^2}\int\varepsilon_1 dt=0; \tag{36b}$$

$$\vartheta=\frac{1}{6a\varepsilon_1 u_1^2}\int\varepsilon_1 dt. \tag{36c}$$

Using (31), equations (36) can be readily solved by putting, as before,

$$\varphi_1=(\overline{C}_1\xi t/a^2)\,u_1^2;\quad \varphi_2=(\overline{C}_2\xi t/a^2)\,\varepsilon_1,$$

upon substitution we obtain

$$\varphi_1\sim-(\xi t/4a^2)\,u_1^2;\quad \varphi_2\sim(\xi t/2a^2)\,\varepsilon_1. \tag{35a}$$

The condition $t\xi/a^2 \lesssim 1$ is close to the analogous condition derived by L. D. Landau, but with the substantial difference that in Ref. 1 the condition is stated in the form of a strong inequality. The formal similarity is not surprising since both derivations employed the relationship $u^2 \sim t/\xi$, which, as we verified, is approximately satisfied whether $y^2 \sim \rho^2/3$ or $y^2 \gg \rho^2/3$.

4. Let us next examine the temperature limits of the applicability of the unidimensional motion. First of all, let us express the parameter $y = 1/4 \ln(\epsilon/\epsilon_0)$ in terms of the enrgy E_0 of the incident particle. The initial energy density is

$$\varepsilon_0=\overline{E}_0/V, \tag{36}$$

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*We note that the exponent $(2-\sqrt{3})/2$ is considerably smaller than unity.

where $\bar{E}_0 = \sqrt{2E_0}$ is the total energy in the center-of-mass system; let us also assume, in agreement with Fermi[2], $V = (4\pi/3\mu^3)\sqrt{2/E_0}$; substituting the expression for V into (36) we get

$$\varepsilon_0 = (3/4\pi)\,E_0\mu^3. \tag{37}$$

From simple thermodynamic considerations (see, for example, Ref. 4 or 7) we can ascertain that at a temperature T the boson energy density is

$$\varepsilon = (6{,}5/2\pi^2)\,a_1 T^4, \tag{38}$$

and the fermion energy density is

$$\varepsilon = (5{,}7/2\pi^2)\,a_2 T^4, \tag{39}$$

where a_1 and a_2 are the numbers of the internal degrees of freedom of the particles; $a_1 = 3$ for pi-mesons and $a_2 = 8$ for nucleon-antinucleon pairs. Strictly speaking, Eqs. (38) and (39) are valid if T is many times greater than the rest mass $\mathfrak{M}$ of the particles under consideration. However, by applying the rigorous expressions for the energy density[4] it can be shown that even if $T \sim \mathfrak{M}$ the values obtained from (38) and (39) differ from the true ones by approximately 10%. Hereinafter we shall restrict ourselves to pi-mesons and to temperatures $T \geq \mu$, using therefore Eq. (38). Equations (38) and (39) yield

$$y = {}^1\!/_4 \ln (4T^4/E_0\mu^3). \tag{40}$$

Let us consider next the leading edge. According to (12)

$$t\xi/a^2 = (2/E_0)\,e^{-4y} = \mu^3/2T^4. \tag{41}$$

We thus obtain $t\xi/a^2 = 1$ at $T \sim 1.3\,\mu$ and $t\xi/a^2 \sim 3$ at $T = \mu$. Equation (37) indicates that as far as the leading edge is concerned, the motion of the liquid can be said to be quasi-unidimensional at temperatures up to $T \geq \mu$. The quantity $t\xi/a^2$ is greatly dependent on the temperature, and at $T_k < \mu$ the lateral breakup would therefore influence the energy characteristics quite substantially.

It is interesting to note that Eq. (41) does not depend on the energy. This shows that the diminishing breakup angles and the longer path covered by the liquid with increasing E_0 compensate each other.

Consider next the region of the slowest particles. Using (31) and (40), we readily obtain for this region

$$t\xi/a^2 = (1/4\cdot 3^{3/2})\,E_0^{1/2}\mu^{9/2}T^{-6}. \tag{42}$$

For an energy $E_0 = 10^5$, we obtain $t\xi/a^2 = 1$ if $T \sim 2.5\,\mu$, but $t\xi/a^2 \sim 5$ if $T \sim 2\mu$.

Apparently the quantity $t\xi$ increases continuously with diminishing ρ. Although this statement has not been rigorously proven, the following estimates make its truth quite likely. We examined

$$\frac{\partial (t\xi)}{\partial \rho} \sim \frac{\partial}{\partial \rho}\left[\left(\frac{\partial \chi}{\partial y}\right)^2 - \left(\frac{\partial \chi}{\partial \rho}\right)^2\right]$$

for the leading edge [see Eq. (10)] and for the logarithmic approximations (25) and (26). In both cases we found

$$\partial (t\xi)/\partial \rho < 0.$$

Consequently, the lower the energy of the escaping particles, the more pronounced the lateral breakup, with the lowest temperature at which the quasi-unidimensional solution is valid in all regions being given approximately by Eq. (42).

In conclusion, let us tabulate the values of $\xi t/a^2$ for various values of E_0 and T, numerically determined from the exact Eqs. (3) to (6).

$E_0 = 3\cdot 10^5;\ T = 2\mu$		$E_0 = 4\cdot 10^4;\ T = 1{,}5\mu$		$E_0 = 10^3;\ T = \mu$		$E_0 = 2\cdot 10^4;\ T = \mu$	
u_0	$\xi t/a^2$	u_0	$\xi t/a^2$	u_0	$\xi t/a^2$	u_0	$\xi t/a^2$
49	0.2	11	0.8	11	3,2	49	3,2
24	0,35	5.5	1,0	5,5	4.4	24	4,5
10	0.7	3.0	1.2	3	4,7	10	8
3	1.2	1.4	1.6	1,4	6,2	3	16
1,4	2.0	1.1	2,0	1,1	8,0	1.4	25

Comparing the data in the last line of the table against those obtained from Eq. (42) we see that the latter apparently gives a value that is approximately 2–3 times too high. The motion thus is quasi-unidimensional at $T \sim 1.5\mu$ to 2μ. It is interesting to check the validity of the quasi-unidimensional approximation at $T \sim \mu$. As can be seen from the Table [cf. also (41)], the regions adjacent to the leading edge satisfy approximately the criterion $t\xi / a^2 \sim 1$, but the quasi-unidimensional nature is lost in the region of the slower particles. Nevertheless, a noticeable difference does take place as E_0 changes from 10^3 to 2×10^4. At $E_0 \sim 10^3$ we get for the slow particles $\xi t / a^2 \sim 6$ to 8 and consequently $\varphi_1 \sim u_1^2$ and $\varphi_2 \sim \epsilon_1$ [cf. (35a)]. We can therefore expect the quasi-unidimensional approximation to give the correct order of magnitude for these particles. One cannot expect more in this region, for the condition $u >> 1$ is no longer satisfied here. Consequently, it becomes probable that for $T \sim \mu$ (or $E_0 \sim 10^3$) the quasi-unidimensional approach will also yield greater accuracy than the conical breakup. A somewhat different situation arises at higher energies, where we can have $\varphi_1 >> u_1^2$ and $\varphi_2 >> \epsilon_1$.

5. Let us analyze the temperature dependence* of the energy $\bar{E}_m$ of the fastest particle.[8] The velocity u_m of this particle is determined from the relationship

$$\rho_m = -\sqrt{3}\, y, \tag{43}$$

which is satisfied in the self-modeling motion region (see Ref. 5). Substituting the value of y from Eq. (40) and using $u = e\rho/2$ we get

$$u_m = \tfrac{1}{2}(E_0\mu^3/4)^{\sqrt{3/4}}\, T^{-\sqrt{3}}. \tag{44}$$

To obtain E_m (in center-of-mass system) we shall assume that the particles bound to the element move isotropically in the system. Then, assuming that the secondary particles are mesons, we have

$$\bar{E}_m = 2\beta T_k u_m = \beta(E_0\mu^3/4)^{\sqrt{3/4}}\, T^{-(\sqrt{3}-1)}, \tag{45}$$

where $\beta \sim 2$ (for $T_k \sim \mu$, $\beta = 1.8$).[7]

The transverse components can be neglected in the transition to the laboratory system, and therefore

$$\bar{E}_m = 2^{-\frac{1}{2}(\sqrt{3}-1)}\beta E^{(2+\sqrt{3})/4}\mu^{3\sqrt{3/4}} / T^{(\sqrt{3}-1)}. \tag{46}$$

We thus obtain for $E_0 = 10^3$

$$\bar{E}_m \sim 0.35 E_0 \quad (T_k = \mu);$$

$$\bar{E}_m \sim 0.25 E_0 \quad (T_k = 2\mu);$$

$$\bar{E}_m = 0.1 E_0 \quad (T_k = 3\mu).$$

It follows from an analysis[9] of experimental data[10,11] that at these energies one of the particles carries off a considerable fraction ($\gtrsim 1/2$) of the primary energy. One can therefore expect within the framework of the hydrodynamic theory to have $T_k \leq \mu$*. This conclusion agrees with the deduction made essentially by Belen'kii[12] who compared the dependence of the fraction of energy carried off by heavy particles on T_k with Grigorov's[13] and Zatsepin's[9] experimental data. The relationship $T_k \lesssim \mu$ leads to an important deduction. Very simple estimates[4] show that if $T_k = 1.5\mu$ and $\sigma = (1/\mu)^2/3$, the mean free path is approximately three times greater than the linear dimensions of the system, $1/\mu$. In this case the free breakup should therefore occur even at $T_k \sim 1.5\mu$, contradicting the condition $T_k \lesssim \mu$. One can conclude from this that at $T_k \sim \mu$ the interaction cross section of the secondary particles is of the order of $1/\mu^2$

In conclusion, we must emphasize the very preliminary character of this deduction. It is based essentially on the study of the behavior of the fastest particle, a study that cannot be carried out conclusively within the framework of pure hydrodynamics. One should rather examine the entire leading edge as a whole (cf. Ref. 7). The situation becomes even more complicated by the fact that the experimental data on hand indicate[9,13] that as a rule the fastest particle is a nucleon. If this fact is confirmed at sufficiently high energies ($\geq 10^{12}$ ev), the only ones for which the hydrodynamic treatment can be deemed valid, it will become necessary to resort to nonstatistical factors to explain this fact. A possible way out, within the spirit of the hydrodynamic conception, is to refine the initial conditions of the problem (to assume, for example, that the first particle escaping from the volume is a nucleon, which acts so to speak as a "piston" for the entire remaining system*).

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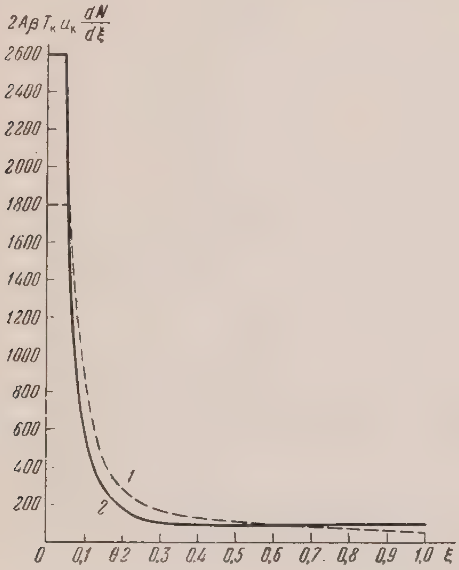
*The behavior of the fastest particle at $T \sim \mu$ was studied previously by Gerasimova and Chernavskii.[8]

*Although some discrepancy between the experimental and theoretical data remains at $T_k = \mu$ there are many factors (thermal motion[7], presence of heavy particles[12]) that should tend to decrease this discrepancy.



$1 - T_k = 3{,}0\ \mu,\ 2 - T_k = 1{,}5\ \mu$

6. In this section, let us consider the energy distribution. Since the number dN of particles in an element of liquid is proportional to the entropy dS contained in the element, we can write

$$dN \sim s_k u_k R_k^2 d\xi, \qquad (47)$$

where s_k and R_k are the entropy density and the transverse dimensions of the element at the instant of free breakup.

Inasmuch as $s_k \sim T_k^3 = \text{const.}$, we have

$$dN = A u_k R_k^2 d\xi, \qquad (48)$$

$A = \text{const}$; $R_k = 1 + t_k^{\;2} / a^2 u_k^2$; we can put approximately

$$R_k = 1 + t_k \xi_k / a_k.$$

The constant A is determined from the equality

$$2A\beta T_k \int u_k dN = \bar{E}_0/2. \qquad (49)$$

The energy distribution thus depends on the function $u_k(\xi_k)$. Approximate estimates can be made from equations (7), (8), (11), (25), (26), and (31).

By way of illustration, the figure shows the numerically-computed energy distributions at $E_0 = 2 \times 10^4$ and at temperatures $T_k = 1.5\mu$ and $T_k = 3\mu$.

In conclusion, the author acknowledges his debt to L. D. Landau, S. Z. Belen'kii, and I. M. Khalatnikov for discussing individual problems touched upon in this investigation.

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Translated by J. G. Adashko
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*The value of $\bar{E}_m$ depends also on the volume V, of which only the order of magnitude is known. Using (37), (43), and (45), it is easy to show that $\bar{E}_m \sim V^{-\sqrt{3/4}}$.

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Thermal Radiation of Good Conductors

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Radiation from good conductors is examined by the methods of the electrodynamical theory of thermal fluctuations. In the first part of the work, radiation in the wave zone is found, with particular attention devoted to the limiting cases of very short and very long waves. Radiation from bodies with surface anistropy is examined. In the second part of the work, the fluctuating field in the neighborhood of conducting surfaces is considered: near a metallic plane, at the focus of a parabolic mirror, and at the center of a spherical mirror. Fluctuating surface charges are calculated.

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1. THERMAL FIELD IN THE WAVE ZONE

THE analysis of thermal radiation from heated bodies can be considered as a problem of phenomenological electrodynamics if, following Rytov[1,2], random extraneous fields, or equivalently, random extraneous currents with zero radius of correlation, are introduced as the source of this radiation. As has been shown by the author[2], the electrodynamic reciprocity theorem allows a substantial simplification of the problem, reducing it to quadratures, if the solution to the corresponding subsidiary diffraction problem is known. In particular, the spectral density P_ω of the flow of energy of the fluctuating field in the wave zone, per unit solid angle, is connected with the effective transversality of the absorption for a plane wave with the same polarization as the radiation field by the formula

$$P_\omega = (\Theta / 2\pi) A / \lambda^2, \qquad (1.1)$$

where $\Theta = kT$ is the temperature of the body in energy units (we limit ourselves to the classical region of frequencies $\hbar\omega \ll \Theta$), and λ is the wavelength. The direction of the quantity P_ω is the direction of the infinitely distant source of the incident plane wave.

The effective transversality of the absorption is

$$A = \frac{Q_0}{(c/8\pi)\,|E_{\text{inc}}|^2} = \frac{Q_0}{(c/8\pi)\,|H_{\text{inc}}|^2}, \qquad (1.2)$$

where Q_0 are the thermal losses of the diffracted field in the body under consideration.

For good conductors, the diffracted field inside the body differs from zero only in a thin surface layer, whose thickness we may consider small with respect to all dimensions of the body and wavelength. In this case, Joule losses may be calculated by the formulas of the theory of the strong skin effect[4]: the heat arriving at an element of the surface of the body dS is

$$dQ_0 = (c/8\pi)\,\eta\,|H_0|^2\,dS; \quad \eta = \sqrt{\mu\omega/8\pi\sigma}, \qquad (1.3)$$

where H_0 is the tangential component of the magnetic vector of the diffracted field on the surface of the body, σ is the conductivity, and μ is the magnetic permeability. To a first approximation, H_0 is the same as it would be at the surface of an ideally conducting body.

Thus, for good conductors, the spectral density of radiation of interest to us can be represented in the form

$$P_\omega = \frac{\Theta}{2\pi}\,\frac{\eta}{\lambda^2\,|H_{\text{inc}}|^2}\oint |H_0|^2\,dS = \frac{\Theta\eta S}{2\pi\lambda^2}\,G. \qquad (1.4)$$

where S is the surface area of the body, and

$$G = \oint |H_0|^2\,dS \,/\, |H_{\text{inc}}|^2 S \qquad (1.5)$$

is a dimensionless function of direction and polarization, generally of order 1.

Integrating Eq. (1.4) over both polarizations and all directions, we obtain the spectral density J_ω of the total radiation

$$J_\omega = \frac{\Theta\eta S}{2\pi\lambda^2}\int (G_1 + G_2)\,d\Omega, \qquad (1.6)$$

where $d\Omega$ is the element of solid angle, and G_1 and G_2 correspond to the two different polarizations. In the future, in considering solids of revolution and plane laminae, G will be denoted by $G_\parallel$ for a wave whose electric vector lies in the meridian plane (or plane of incidence), and by $G_\perp$ for a wave whose magnetic vector lies in the meridean plane.

Exact solutions of diffraction problems (for example, for a sphere or disc) are represented by functional series, the substitution of which into Eq. (1.5) gives, after completing the quadratures, a certain series for the desired function $\bar{G}$. We will not consider these exact solutions here, but will limit ourselves to those cases in which all the dimensions of the body are either great or small with respect to the wavelength. Then approximate solutions of the diffraction problem can be used (geometrical optics, quasi-stationary approximation), yielding finite algebraic expressions for G.

1. Shortwave Radiation

If the wavelength λ is small compared to all dimensions (including radii of curvature) of an ideally conducting convex body, then in the geometrical optics approximation, the diffraction field H_0 is zero on the shaded part of the surface of the body, and on the illuminated part is determined by the reflection formula

$$\mathbf{H}_0 = 2\,[\mathbf{h} - \mathbf{n}(\mathbf{n}\cdot\mathbf{h})]\,H_{\text{inc}} \tag{1.7}$$

where n is a unit vector normal to the surface of the body, and h is the base vector of the magnetic vector of the incident wave. Substituting (1.7) into the general formula (1.5), we obtain

$$G = \frac{4}{S}\int [1 - (\mathbf{n}\cdot\mathbf{h})^2]\,dS, \tag{1.8}$$

where the integral is taken only over the illuminated portion of the surface of the body. For bodies having a center of symmetry, the integral over the shaded portion is evidently equal to the integral over the illuminated portion, so that the formula (1.8) for such bodies can be written in the form

$$G = \frac{2}{S}\oint [1 - (\mathbf{n}\cdot\mathbf{h})^2]\,dS = 2\,[1 - \overline{(\mathbf{n}\cdot\mathbf{h})^2}],$$

where the bar denotes an average over the surface of the body. Let us consider two examples:

a) *Plane thin Lamina* (Fig. 1). For a parallel-polarized wave, the vector h is parallel to the lamina, and $\mathbf{n}\cdot\mathbf{h} = 0$. For a perpendicular-polarized wave, $\mathbf{n}\cdot\mathbf{h} = \sin\vartheta$. Consequently,

$$G_{\parallel} = 2;\quad G_{\perp} = 2\cos^2\vartheta. \tag{1.9}$$

For $\vartheta = 0$, as could be expected, $G_{\parallel} = G_{\perp}$; for $\vartheta = \pi/2$, $G_{\perp} = 0$ and the thermal radiation is linearly polarized.

b) *Solid of Revolution With a Center of Symmetry* (Fig. 2). Let the origin of the coordinates coincide with the center of symmetry, with the z-axis along the axis of revolution of the body. The direction cosines of the normal vector n are denoted as usual by α, β, γ, and the orientation of h by h_1, h_2, h_3. In view of symmetry, evidently,

$$\overline{\alpha\beta} = \overline{\beta\gamma} = \overline{\gamma\alpha} = 0;\quad \overline{\alpha^2} = \overline{\beta^2} = {}^1\!/_2\,(1 - \overline{\gamma^2}),$$

so that

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1\!/_2\,(1 - \overline{\gamma^2})\,(h_1^2 + h_2^2) + \overline{\gamma^2}h_3^2.$$

For a parallel-polarized wave

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1\!/_2\,(1 - \overline{\gamma^2}).$$

For a perpendicular-polarized wave

$$\overline{(\mathbf{n}\cdot\mathbf{h})^2} = {}^1\!/_2\,(1 - \overline{\gamma^2})\cos^2\vartheta + \overline{\gamma^2}\sin^2\vartheta.$$

Thus, for solids of revolution with a center of symmetry

$$G_{\parallel} = 1 + \overline{\gamma^2}; \tag{1.10}$$

$$G_{\perp} = 1 + \overline{\gamma^2} + (1 - 3\overline{\gamma^2})\sin^2\vartheta.$$

Just as in the case of the lamina, $G_{\parallel}$ does not depend on the angle ϑ. For a sphere, $\overline{\gamma}^2 = 1/3$, and

$$G_{\parallel} = G_{\perp} = {}^4\!/_3. \tag{1.11}$$

For a thin disc, $\overline{\gamma}^2 \approx 1$, and

$$G_{\parallel} = 2;\quad G_{\perp} = 2\cos^2\vartheta,$$

i.e., we obtain the result found earlier. For a thin extended needle, $\overline{\gamma}^2 \approx 0$, and

$$G_{\parallel} = 1;\quad G_{\perp} = 1 + \sin^2\vartheta.$$

Generally, for an oblate convex solid of revolution, it is evident that $\overline{\gamma}^2 > 1/3$, and consequently, according to (1.10), $G_{\perp} \leq G_{\parallel}$, and $G_{\perp}$ is maximum in the direction of the axis of revolution $\vartheta = 0$. But for a prolate convex solid of revolution, $\overline{\gamma}^2 < 1/3$, $G_{\perp} \geq G_{\parallel}$, and $G_{\perp}$ is maximum in the equatorial plane $\vartheta = \pi/2$.

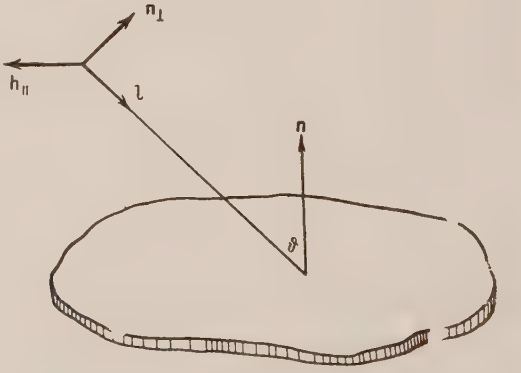

FIG. 1

In particular, for an oblate spheroid of eccentricity e

$$\overline{\gamma^2} = \frac{(\mathrm{ch}\,\chi + 1)(\mathrm{sh}\,\chi - \chi)}{(\mathrm{ch}\,\chi - 1)(\mathrm{sh}\,\chi + \chi)}\ ;\ \chi = 2\,\mathrm{arth}\,e \qquad (1.12)$$

and for a prolate spheroid of eccentricity e

$$\overline{\gamma^2} = \frac{(1+\cos\varphi)(\varphi-\sin\varphi)}{(1-\cos\varphi)(\varphi+\sin\varphi)}\ ;\ \varphi = 2\arcsin e. \qquad (1.13)$$

A graph of $\overline{\gamma}^2$ versus e, based on Eqs. (1.12) and (1.13), is given in Fig. (3). Using this graph and the general formula (1.10), the shortwave thermal radiation can be calculated for any metallic spheroid.

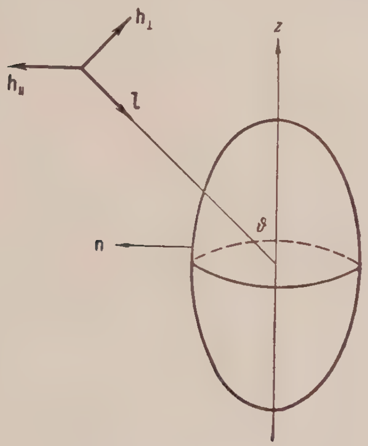

FIG. 2

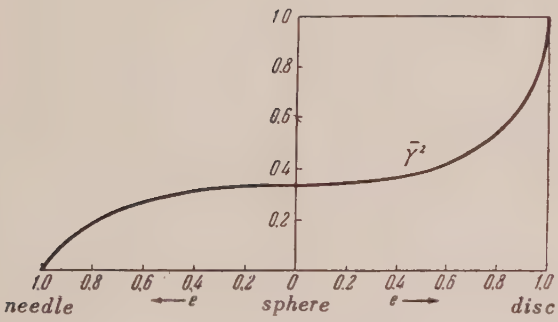

FIG. 3

Substituting (1.9) or (1.10) into the general formula (1.6), we easily find that in both cases the spectral density of the total radiation is

$$I_\omega = 16\,\Theta\gamma S/3\lambda^2. \qquad (1.14)$$

We will now show that formula (1.14) is valid for an arbitrary convex body. We note that this could be expected beforehand, since in the geometric optics approximation, the usual formulation of Kirchoff's law is valid, and the total thermal radiation does not depend on the form of the body, but only on its surface area.

Adding Eqs. (1.8) for the two possible mutually perpendicular polarizations $\mathbf{h}_1$ and $\mathbf{h}_2$ and noting that $(\mathbf{n}\cdot\mathbf{h}_1)^2 + (\mathbf{n}\cdot\mathbf{h}_2)^2 = 1 - (\mathbf{n}\cdot\mathbf{l})^2$, where $\mathbf{l}$ is the base vector of the wave vector of the incident wave, we obtain for the summed factor $G_t = G_1 + G_2$ the following expression:

$$G_t = \frac{4}{S}\int [1 + (\mathbf{n}\cdot\mathbf{l})^2]\,dS.$$

We will integrate this equation over all directions. To each direction may be compared its direct opposite, for which an illuminated part of the surface appears as a shaded part for the original direction. Therefore,

$$\int G_t\,d\Omega = \frac{2}{S}\int\oint [1 + (\mathbf{n}\cdot\mathbf{l})^2]\,dS\,d\Omega.$$

But for arbitrary $\mathbf{n}$,

$$\int (\mathbf{n}\cdot\mathbf{l})^2\,d\Omega = 4\pi/3,$$

so that

$$\int G_t\,d\Omega = 32\pi/3. \qquad (1.15)$$

Substituting (1.15) into (1.6), we again arrive at Eq. (1.14) for the total radiation of an arbitrary convex metallic body in the geometrical optics approximation.

3. Longwave Radiation

If the wavelength is large compared to the dimensions of the body, we can confine ourselves to a consideration of the diffraction problem in the quasi-stationary approximation. In the quasi-stationary approximation, the magnetic vector is $\mathrm{H}_0 = \mathrm{H}_{pot}$, where H_{pot} is found from the solution of the magnetostatic problem of an ideally conducting body placed in a homogeneous external magnetic field H_{inc}. Actually, the vortical addition H_{vort}, generated by the changing electric field, is determined by the equation $\mathrm{curl}\,\mathrm{H}_{vort} = (2\pi i/\lambda)\mathrm{E}$. But considering order of magnitude, $|\mathrm{curl}\ \mathrm{H}_{vort}| \sim (1/a)H_{vort}$ (a is a characteristic dimension of the body, $a \ll \lambda$) and consequently, $H_{vort} \sim (a/\lambda)E \sim (a/\lambda)H_{pot}$, since $H_{pot} \sim H_{inc} \sim E$. Thus in the quasi-stationary approximation, thermal losses are due to the magnetic vector* of the incident wave.

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* Taking account of the electric vector of the incident wave is essential only for very thin rods (wire antennae)having resonance properties. Thermal radiation of such rods has been considered by Rytov and the author^5.

A magnetostatic problem for a body with zero magnetic permeability corresponds formally to an ideally conducting body on the surface of which the normal component of the magnetic vector is zero. We note that when the body possesses three mutually perpendicular planes of symmetry, it is sufficient to find the factor G only for the three principal axes. Let $\mathbf{H}_{inc} = \mathbf{h} = h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k}$ where $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, are the base vectors of the principal axes. Clearly,

$$\mathbf{H}_0 = h_1\mathbf{H}_{01} + h_2\mathbf{H}_{02} + h_3\mathbf{H}_{03},$$

where, for example, $\mathbf{H}_{01}$ is the magnetic field corresponding to $\mathbf{H}_{inc} = \mathbf{i}$. In view of the symmetry of the body, integrals of the form

$$\oint \mathbf{H}_{01}\mathbf{H}_{02}\,dS = 0,$$

and therefore, for arbitrary direction $\mathbf{h}$, the function G is

$$G = h_1^2 G_1 + h_2^2 G_2 + h_3^2 G_3. \tag{1.16}$$

As an example, we consider a body having the form of an ellipsoid with semiaxes a, b, c $(a \geq b \geq c)$. Substituting the solution of the magnetostatic problem for an ellipsoid with $\mu = 0$ (see for example Ref. 6, where the detailed solution of the equivalent electrostatic problem is given), we obtain after straightforward but somewhat laborious calculations, the following expression for G_1:

$$SG_1 = \frac{2}{(1-M_1)^2}\,\frac{a^2}{(a^2-b^2)(a^2-c^2)} \times\left\{\int_{c^2}^{b^2}\sqrt{\frac{(b^2-u)(u-c^2)}{u(a^2-u)}}\,du \times\int_{b^2}^{a^2}\sqrt{\frac{v(a^2-v)}{(v-b^2)(v-c^2)}}\,dv + \int_{c^2}^{b^2}\sqrt{\frac{u(a^2-u)}{(b^2-u)(u-c^2)}}\,du \times\int_{b^2}^{a^2}\sqrt{\frac{(v-b^2)(v-c^2)}{v(a^2-v)}}\,dv\right\}, \tag{1.17}$$

where

$$M_1 = \frac{abc}{2}\int_0^\infty ds/(a^2+s) \times\sqrt{(a^2+s)(b^2+s)(c^2+s)} \tag{1.18}$$

Formulas for G_2 and G_3 are obtained from (1.17) and (1.18) by cyclic permutation in the expressions under the integrals and in the factor outside the integrals, with a subsequent change of sign if negative expressions are obtained during such permutations. We note that $M_1 + M_2 + M_3 = 1$. We will not write here the representations of the right hand side of (1.17), (1.18), as well as the area of the ellipsoid S by elliptic integrals, but will limit ourselves to the case of an ellipsoid of revolution for which the quantities of interest to us are expressed by elementary functions. Omitting the intermediate transformations, we present the final formulas.

Oblate spheroid $(a = b \geq c)$ of eccentricity e:

$$\begin{aligned} G_3 &= G_p \\ &= 2(1-M)^{-2}(\chi\operatorname{ch}\chi - \operatorname{sh}\chi) \\ &\quad\div(\operatorname{ch}\chi - 1)(\operatorname{sh}\chi + \chi), \\ G_1 &= G_2 = G_E \\ &= 4(1+M)^{-2}(\operatorname{ch}\chi\operatorname{sh}\chi - \chi) \\ &\quad\div(\operatorname{ch}\chi - 1)(\operatorname{sh}\chi + \chi), \\ M &= M_3 = \operatorname{cth}^2\frac{\chi}{2}\left(1 - \frac{\operatorname{arctg}\operatorname{sh}(\chi/2)}{\operatorname{sh}(\chi/2)}\right); \end{aligned} \tag{1.19}$$

where χ is associated with e by formula (1.12).

Prolate spheroid $(a \geq b = c)$:

$$\begin{aligned} G_1 &= G_p \\ &= 2(1-M)^{-2}(\sin\varphi - \varphi\cos\varphi) \\ &\quad\div(1-\cos\varphi)(\varphi + \sin\varphi); \\ G_2 &= G_3 = G_E \\ &= 4(1+M)^{-2}(\varphi - \cos\varphi\sin\varphi)/(1-\cos\varphi) \\ &\quad\times(\varphi + \sin\varphi); \\ M &= M_1 = \operatorname{ctg}^2\frac{\varphi}{2}\left(\frac{\operatorname{arth}\sin(\varphi/2)}{\sin(\varphi/2)} - 1\right); \end{aligned} \tag{1.20}$$

where φ is associated with e by formula (1.13). The indices "p" and "e" in Eqs. (1.19) and (1.20) denote "pole" and "equator," respectively.

For solids of revolution, the general formula (1.16) can be written in the form

$$G = h_p^2 G_P + h_E^2 G_E,$$

where h_p is the projection of $\mathbf{h}$ on the axis of revolution, and h_e is the projection on the equatorial plane. For parallel and perpendicularly polarized waves, respectively,

$$G_{\parallel} = G_E;\ G_{\perp} = \sin^2\vartheta G_P + \cos^2\vartheta G_E. \quad (1.21)$$

For a sphere ($e = 0$), Eqs. (1.19) and (1.20) give $G_p = G_e = 3/2$, so that*

$$G_{\parallel} = G_{\perp} = {}^3/_2. \qquad (1.22)$$

For a disc ($e \to 1$), it follows from (1.19) that

$$G_P = \frac{8}{\pi^2}\left(\operatorname{arth} e - \frac{1}{2}\right) \approx \frac{8}{\pi^2}\left(\ln\frac{2a}{c} - \frac{1}{2}\right);$$

$$G_E = 1$$

and

$$G_{\parallel} = 1;\ G_{\perp} = \frac{8}{\pi^2}\left(\ln\frac{2a}{c} - \frac{1}{2}\right)\sin^2\vartheta + \cos^2\vartheta.$$

For a needle ($e \to 1$), formula (1.20) gives $G_p = 1$ and $G_e = 2$, so that

$$G_{\parallel} = 2;\ G_{\perp} = 1 + \cos^2\vartheta. \qquad (1.23)$$

The dependence of G_p and G_e on the eccentricity e, calculated by formulas (1.19) and (1.20), is shown in Fig. (4). Using these curves, the long wave thermal radiation of an arbitrary metallic spheroid can be calculated by Eq. (1.21).

For an oblate spheroid, $G_p > G_e$ and on the basis of Eq. (1.21), $G_{\perp}$ is maximum in the equatorial plane $\vartheta = \pi/2$. For a prolate spheroid, $G_p < G_e$, and $G_{\perp}$ is maximum in the direction of the axis of revolution $\vartheta = 0$. In the case of short wave radiation, the situation is opposite as was shown in Section 1.

In Section 1 we established that the short wave radiation of a thin plane lamina of arbitrary form possesses circular symmetry with respect to an axis perpendicular to the lamina. It is easily seen that long wave radiation possesses the same symmetry. In fact, a thin lamina with $\mu = 0$, placed in a homogeneous magnetic field does not disturb the field. Therefore, for coordinate axes x and y chosen in the plane of the lamina, $G_1 = G_2 = 1$, whence follows circular symmetry of the radiation. For the third axis z perpendicular to the lamina, the factor G_3 is of logarithmic magnitude. For example, in the case of a strongly prolate elliptical disc ($a >> b >> c$), Eqs. (1.17) and (1.18) give $G_1 = G_2 = 1$; $G_3 = \ln(4b/c) - 1$.

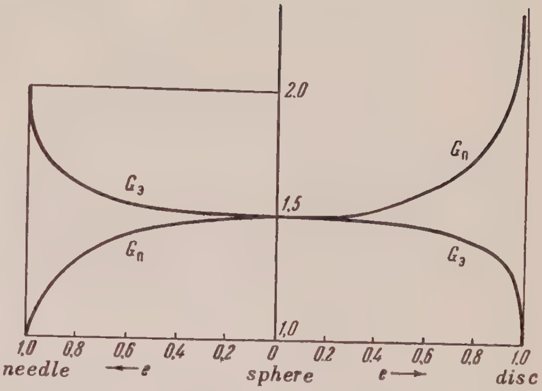

FIG. 4

Thus, for long wave radiation of any thin plane lamina

$$G_{\parallel} = 1;\ G_{\perp} = \Lambda\sin^2\vartheta + \cos^2\vartheta,$$

where the factor Λ is of the order of the logarithm of the ratio of the width of the lamina to its thickness.

At the beginning of this section it was mentioned that very thin rodlike conductors (wire antennae) possess resonance properties, thanks to which the Joule losses due to the electric vector of the incident wave may become commensurate (even in the case of small bodies) with the losses associated with the magnetic vector. To these "electrical losses" correspond the thermal antenna radiation considered in Ref. (5). As was shown in Ref. (5), for small bodies, the antenna radiation has a dipole character: it is || -polarized and is proportional to $\sin^2\vartheta$, where ϑ is the angle between the rod and the direction of radiation. But according to the first of Eqs. (1.23), the "magnetic" radiation considered above of the same polarization does not depend on the angle ϑ. Let us compare the spectral densities of both radiations. For example, let the rod have the form of a strongly prolate spheroid $a >> b = c$. The spectral density of thermal "antenna" radiation of this spheroid for long waves ($\lambda >> a$) was calculated in Ref. (5), and in the notation of the present article is

$$J_{\text{ant}} = \frac{\pi^4 a^5}{c\lambda^4}\left(\ln\frac{a}{c}\right)^2\Theta\eta.$$

The spectral of the "magnetic" radiation of interest to us is ||-polarized according to Eq. (1.6), and the first of Eqs. (1.23) gives

$$J_{\text{mag}} = 4S\lambda^{-2}\Theta\eta = 4\pi^2 ac\lambda^{-2}\Theta\eta.$$

Thus, for a small thin needle,

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* The curve of the transition from (1.22) to (1.11) is given in Ref. (2).

$$\frac{J^{\text{ant}}}{J^{\text{mag}}} = \left(\frac{\pi}{2}\frac{a^2}{c\lambda}\ln\frac{a}{c}\right)^2 .$$

3. Radiation of Bodies with Surface Anisotropy

In radio technology and radiophysics, more and more use is being made of conductors with anisotropic surfaces, characterized by a surface impedance tensor (see, for example, Ref. 7). Joule losses in such conductors are determined by the formula

$$dQ_0 = (c/8\pi)\,(\eta_u\,|H_{0u}|^2 + \eta_v\,|H_{0v}|^2)\,dS, \tag{1.24}$$

clearly a generalization of Eq. (1.3). u and v are orthogonal curvilinear coordinates on the surface with respect to which the surface impedance tensor is diagonal, and

$$\eta_u = \sqrt{\omega\mu_u/8\pi\sigma_v}; \quad \eta_v = \sqrt{\omega\mu_v/8\pi\sigma_u},$$

where σ_u, σ_v, μ_u, μ_v are the effective quantities corresponding to each type of laminated or similar structure.

Introducing the mean value η,

$$\eta_u = (1+\nu)\,\eta; \quad \eta_v = (1-\nu)\,\eta; \quad -1 < \nu < 1$$

and substituting (1.24) into the general formulas (1.2) and (1.1), we again obtain expression (1.4) for the spectral density of radiation, with the factor G given by

$$G = \oint \{(1+\nu)\,|H_{0u}|^2 + (1-\nu)\,|H_{0v}|^2\}\,dS\,/\,|H_{\text{inc}}|^2\,S. \tag{1.25}$$

As an example, we consider the radiation of a sphere for which the lines u = const are parallels and the lines v = const are meridians. If the surface of the sphere is isotropic ($\nu = 0$), then $G_\parallel = G_\perp = G_{\text{iso}}$, where $G_{\text{iso}} = 4/3$ for short waves and $3/2$ for long waves. With $\nu \neq 0$, a straightforward calculation based on Eq. (1.25) gives the same result in both the geometrical optics approximation and the quasi-stationary approximation:

$$G_\parallel = \frac{2-\nu}{2}\,G_{\text{iso}}; \quad G_\perp = \left(\frac{2-\nu}{2} + \frac{3\nu}{2}\sin^2\vartheta\right)G_{\text{iso}}$$

The agreement of the results is explained by the fact that on the illuminated part of the surface of the sphere, the magnetic field H_0 has the same geometry in both the geometrical optics and the quasi-stationary approximations.

With $\eta_u > \eta_v$ (as would be the case for example if a sphere was completely wound with bare wire, the windings of which coincided with the parallels), $\nu > 0$, and the radiation is maximum in the equatorial plane $\theta = \pi/2$. In particular, with $\eta_u \gg \eta_v$ ($\nu \approx 1$)

$$G_\parallel = {}^1\!/_2\,G_{\text{iso}}; \quad G_\perp = {}^1\!/_2\,(1 + 3\sin^2\vartheta)\,G_{\text{iso}}.$$

In the opposite case $\eta_u < \eta_v$ the radiation will have a maximum in the direction of the axis of symmetry of the sphere, $\vartheta = 0$. In particular, with $\eta_u \ll \eta_v$ ($\nu \approx -1$)

$$G_\parallel = {}^3\!/_2 G_{\text{iso}}, \quad G_\perp = {}^3\!/_2\cos^2\vartheta\, G_{\text{iso}}.$$

As a second example we consider the radiation of a thin plane anisotropic lamina, for which the coordinates u and v are the rectangular coordinates x and y. Let ϑ and φ be the polar angles of the wave vector of the incident wave. Then in the geometrical optics approximation, we obtain after a straightforward calculation, in place of Eq. (1.9),

$$G_\parallel = 2\,(1 - 2\nu\cos 2\varphi);$$
$$G_\perp = 2\cos^2\vartheta\,(1 + 2\nu\cos 2\varphi).$$

But in the quasi-stationary approximation, we will have

$$G_\parallel = 1 - 2\nu\cos 2\varphi;$$
$$G_\perp = \Lambda\sin^2\vartheta + \cos 2\vartheta\,(1 + 2\nu\cos 2\varphi).$$

2. FLUCTUATING FIELD NEAR HIGHLY CONDUCTING SURFACES

In the first part of this article, the thermal radiation of good conductors was considered in the wave zone. However, the fluctuating field near radiating bodies, at distances commensurable with or smaller than the dimensions of these bodies, also has practical interest. As was shown in Ref. (3), the problem of finding the fluctuating field at any point outside the radiating body may be reduced to quadratures by use of the electrodynamic theorem of reciprocity, if the diffracted field created by an elementary source situated at this point is known. Thus, for a uniformly heated body at the temperature Θ, the mean square spectral density of any component of the electric vector is*

$$\overline{|\mathbf{E}|^2} = 2\Theta Q_0/\pi\,|\mathbf{F}|^2, \tag{2.1}$$

---

* We are using the same notation as in Refs. 2 and 3. The index ω in the spectral density denotes that it corresponds to the decomposition in the spectrum according to positive frequencies. Wherever fluctuating quantities are mentioned, we have in mind spectral densities.

where $F = \int j_0 dV = i\omega p$ is a quantity characterizing the electric dipole formed by the extraneous electric currents j_0 flowing in the vanishingly small volume of integration τ, and Q_0 represents the thermal losses of the diffracted field created by these dipoles in the body under consideration. The direction of the vector F coincides with the direction of the component of the electric vector of interest to us. An equation similar to Eq. (2.1) is valid for the magnetic vector of the fluctuating field, the only difference being that the source of the diffracted field is a magnetic dipole, formed by extraneous magnetic currents j_0^m, $F = \int j_0^m dV$.

In good conductors, the diffracted field differs from zero only in a thin skin-layer. In the future we will always consider that all dimensions of the body, the wavelength, and the distance from the surface of the body to the point at which the fluctuating field is determined, are great in comparison with the thickness of the skin-layer δ. Then the thermal losses Q_0 may be calculated by the theory of the strong skin effect

$$Q_0 = (c\eta/8\pi) \oint |H_0|^2 dS. \tag{2.2}$$

Substituting (2.2) into (2.1), and introducing as a standard the mean square of the spectral densities of the field vectors in equilibrium radiation

$$\overline{|\mathbf{E}_{\text{eq.}}|^2} = \overline{|\mathbf{H}_{\text{eq.}}|^2} = 2\Theta k^2/\pi c,$$

we obtain

$$\overline{|\mathbf{E}|^2} = {}^1\!/_2\, \eta\, \overline{|\mathbf{E}_{\text{eq.}}|^2}\, e; \tag{2.3}$$

$$\overline{|\mathbf{H}|^2} = {}^1\!/_2\, \eta\, \overline{|\mathbf{H}_{\text{eq.}}|^2}\, h.$$

Here, e and h are dimensionless factors depending on the geometry of the body, the wavelength, and the coordinates of the point at which the fluctuating field is being determined,

$$e = \frac{c^2}{4\pi k^2} \oint |H_0^{\text{el}}|^2 dS; \tag{2.4}$$

$$h = \frac{c^2}{4\pi k^2} \oint |H_0^{\text{mag}}|^2 dS,$$

where H_0^{el} and H_0^{mag} are the diffracted fields of unit ($F = 1$) electric and magnetic dipoles with pertinent orientations. The factor 1/2 is introduced into Eq. (2.3) for convenience: with such normalization, the factor e for the component of the electric vector parallel to a radiating plane tends toward unity at large distances from the plane.

In the following, we will consider only those problems in which the diffracted field may be calculated by reflection formulas. In this case, $H_0 = 2H_{1\,\tan}$, where H_1 is the field of a unit dipole in free space, and Eq. (2.4) may be written in the form

$$e = \frac{c^2}{\pi k^2} \int |[\mathbf{n} \cdot \mathbf{H}_1^{\text{el}}]|^2 dS; \tag{2.5}$$

$$h = \frac{c^2}{\pi k^2} \int |[\mathbf{n} \cdot \mathbf{H}_1^{\text{mag}}]|^2 dS$$

(n is a unit vector normal to the surface). The fields of the unit dipoles in free space, as is well known, are

$$\mathbf{H}_1^{\text{el}} = \frac{ik}{cR}\left(1 - \frac{i}{kR}\right)[\mathbf{f}\rho], \tag{2.6}$$

$$\mathbf{H}_1^{\text{mag}} = \frac{ik}{cR}\left\{\left(1 - \frac{3i}{kR} - \frac{3}{k^2R^2}\right)(\mathbf{f}\rho)\,\rho - \left(1 - \frac{i}{kR} - \frac{1}{k^2R^2}\right)\mathbf{f}\right\}. \tag{2.7}$$

where f is the base vector of the dipole, ρ is the base vector of the radius-vector $\mathbf{R} = R\rho$, and the wave factor e^{-ikR} is omitted.

1. Thermal Field of a Conducting Plane

Let the origin of a cartesian system of coordinates be at the point where a dipole is situated, with the z-axis perpendicular to the radiating plane, and let R, ϑ, φ be spherical coordinates. The coordinate φ is also the polar coordinate on the plane (see Fig. 5). The element of area is $dS = r dr d\varphi = R dR d\varphi$. If the base vector f of an electric dipole is directed along the x-axis, then

$$|[\mathbf{n} \cdot \mathbf{h}_1^{\text{el}}]|^2 = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\cos^2\vartheta = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\frac{z^2}{R^2}. \tag{2.8}$$

Substituting this expression into the general formula (2.5), we obtain, after integration,

$$e_x = 1 + (1/2k^2z^2).$$

For f parallel to the z-axis,

$$|[\mathbf{n} \cdot \mathbf{h}_1^{\text{el}}]|^2 = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\sin^2\vartheta = \frac{k^2}{c^2R^2}\left(1 + \frac{1}{k^2R^2}\right)\frac{R^2 - z^2}{R^2}. \tag{2.9}$$

At large distances the right hand side of Eq. (2.9) decreases as $1/R^2$, so that a direct substitution of (2.9) into the equation for e_z leads to a logarithmically diverging expression. But as is well known, the reflection formulas cease to be valid at large distances; they must be multiplied by the Sommerfeld attenuation function (see for example, Ref. 8), the argument of which is the quantity $(\pi c/2\sigma\mu\lambda^2)r$, called the numerical distance. It was not necessary to take this attenuation into account in order to find e_x, because Eq. (2.8) tends toward zero sufficiently rapidly (as $1/R^4$). But if, as occurs in our case, $H_{\tan}$ falls off only as $1/R$, it becomes necessary to take attenuation into account. To do this, the square of the attenuation function must be introduced under the integral sign. However, we will not do this here, but will limit ourselves to giving a sufficiently accurate approximate calculation; without changing the expression under the integral, we will integrate over the R term giving logarithmic divergence, not up to infinity, but up to the quantity L of the order $L \sim \sigma\mu\lambda^2/c$. Doing this, we obtain

$$e_z = 2\ln\frac{L}{z} + \frac{1}{2k^2z^2}.$$

For a magnetic dipole whose base vector f is parallel to the x-axis,

$$\left|[\mathbf{n}\cdot\mathbf{h}_1^{\text{mag}}]\right|^2 = \frac{k^2}{c^2R^2}\left\{\left(1 - \frac{1}{k^2R^2} + \frac{1}{k^4R^4}\right) - \cos^2\varphi\left[\left(1 - \frac{5}{k^2R^2} - \frac{3}{k^4R^4}\right) + 4\cos^2\vartheta\left(\frac{2}{k^2R^2} + \frac{3}{k^4R^4}\right) - \cos^4\vartheta\left(1 + \frac{3}{k^2R^2} + \frac{9}{k^4R^4}\right)\right]\right\}$$

and after integration ($\cos\vartheta = z/R$)

$$h_x = \ln\frac{L}{z} + \frac{3}{8k^4z^4},$$

where we have again replaced the infinite limit of integration by L for the logarithmic term.

Finally, in the case of a magnetic dipole parallel to the z-axis,

$$\left|[\mathbf{n}\cdot\mathbf{h}_1^{\text{mag}}]\right|^2 = \frac{k^2}{c^2R^2}\left(1 + \frac{3}{k^2R^2} + \frac{9}{k\cdot R^4}\right)\cos^2\vartheta\sin^2\vartheta,$$

$$h_z = {}^1/_2 + 1/2k^2z^2 + 1/4k^4z^4.$$

Since in our case, in view of symmetry, $|\overline{\mathbf{E}}|^2 = 2|\overline{E_x}|^2 + |\overline{E_z}|^2$, the electric energy density of the fluctuating field of a plane is

$$w_\omega^{\text{el}} = \frac{1}{2}\eta w_{\text{eq.}}^{\text{el}}(2e_x + e_z) \tag{2.10}$$

$$= \eta w_{\text{eq}}^{\text{el}}\left(\ln\frac{L}{z} + \frac{3}{4k^2z^2}\right).$$

Similarly,

$$w_\omega^{\text{mag}} = \frac{1}{2}\eta w_{\text{eq.}}^{\text{mag}}(2h_x + h_z) \tag{2.11}$$

$$= \eta w^{\text{mag}}\left(\ln\frac{L}{z} + \frac{1}{4k^2z^2} + \frac{1}{2k\cdot z^4}\right).$$

where

$$w_{\text{eq.}}^{\text{el}} = w_{\text{eq.}}^{\text{mag.}} = {}^1/_2\, w_{\text{eq.}} \qquad w_{\text{eq.}} = \Theta k^2/\pi^2 c,$$

and terms of the order of unity have been omitted on account of the indefiniteness of the quantity L. The total energy of the fluctuating field is

$$w_\omega = w_\omega^{\text{el}} + w_\omega^{\text{mag}} = \eta w_{\text{eq.}}\left(\ln\frac{L}{z} + \frac{1}{2k^2z^2} + \frac{1}{4k\cdot z^4}\right).$$

Let us note once again that all the formulas of the present section are, by the very manner in which they were obtained, valid only for distances z large compared to the thickness of the skin-layer δ; $z \gg \delta$. Moreover, formulas containing the quantity L are valid only under the condition $z \ll L$, which is always realized in practice.

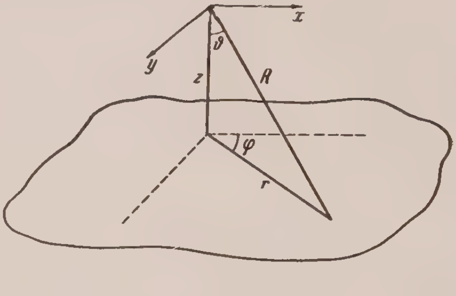

FIG. 5

Comparing Eqs. (2.10) and (2.11), we see that in the neighborhood of the plane, at distances $z \ll \lambda$, the magnetic energy considerably exceeds the electric energy

$$w^{\text{mag}}/w^{\text{el}} = 2/3k^2z^2 = (1/6\pi^2)(\lambda/z)^2,$$

so that $w \approx w^{\text{mag}}$.

Let us now calculate the energy flow vector for thermal radiation from a plane. We will even consider a more general problem: we will find the z-component of the Poynting vector on the axis of a circular disc of large (compared with λ) but finite radius. In view of symmetry,

$$S_{\omega z} = (c/2\pi)\,\text{Re}\,\overline{[\mathbf{E}\cdot\mathbf{H}^*]}_z = (c/\pi)\,\text{Re}\,\overline{E_xH_y^*}. \tag{2.12}$$

As is shown in Ref. (3), the mean value of the product of two components of the fluctuating field, A and B, (in our case, $A = E_x$, $B = H^*_y$) is

$$\overline{AB} = 2\Theta Q_{AB}/\pi F_A F_B, \tag{2.13}$$

where Q_{AB} are the mixed thermal losses A and B of the diffracted field. For good conductors,

$$Q_{AB} = \frac{c\eta}{8\pi}\int[\mathbf{n}\cdot\mathbf{H}_{0A}]\cdot[\mathbf{n}\cdot\mathbf{H}_{0B}]\,dS. \tag{2.14}$$

If H_{0A} and H_{0B} are the fields of unit dipoles, then the factors F_A and F_B in Eq. (2.13) must be omitted. If, moreover, the reflection formulas may be used, then finally

$$\overline{AB} = \frac{c\Theta\eta}{\pi^2}\int [\mathfrak{n}\cdot H_{1A}]\cdot[\mathfrak{n}\cdot H_{1B}]\, dS, \qquad (2.15)$$

where H_{1A} and H_{1B} are the fields of unit dipoles A and B in free space. We note that if B is equal to the complex conjugate of any component of the field, then not the fields H_{0B} and H_{1B} themselves, but their complex conjugates, must be substituted into Eqs. (2.14) or (2.15).

In the case of interest to us, H_{1A} is given by Eq. (2.6) with f = i, and H_{1B} by the complex conjugate of Eq. (2.7) with f = j. Then , as is easily seen,

$$\operatorname{Re}\overline{E_x H_y^*} = \frac{\Theta\eta k^2}{\pi^2 c}\int \cos\vartheta\,(1 - \sin^2\vartheta\sin^2\varphi)\,\frac{dS}{R^2}.$$

Performing the integration and substituting the result into (2.12), we obtain

$$S_{\omega z} = (\Theta\eta k^2/\pi^2)\,[1 - \cos\Phi + {}^1\!/_3\,(1 - \cos^3\Phi)],$$

where Φ is the angle subtended by the radius of the disc. In particular, for an infinite plane ($\Phi = \pi/2$)

$$S_{\omega z} = 4\Theta\eta k^2/3\pi^2 = {}^4\!/_3\,\eta c w_{\text{eq.}}. \qquad (2.16)$$

At normal incidence, the reflection coefficient of a highly conducting plane is $R = 1 - 4\eta$, so that Eq. (2.16), as could be expected, agrees with the expression given by the Kirchhoff theory of thermal radiation.

2. Thermal Field of Metallic Mirrors

Metallic mirrors, used in optics and radio technology as focussing structures, are as a rule large compared with the wavelength. Therefore, to find the fluctuating field at points whose distances from the mirror are also large compared to λ, we may limit ourselves to the geometrical optics approximation. Then only wave terms remain in Eqs. (2.6) and (2.7), and these equations take the form

$$H_1^{\text{el}} = (ik/cR)\,[\mathfrak{f}\rho]; \quad H_1^{\text{mag}} = (ik/cR)\,[[\mathfrak{f}\rho]\,\rho]. \qquad (2.17)$$

The substitution of (2.17) into the general formulas (2.5) for e and h yields the purely geometrical quantities

$$e = \frac{1}{\pi}\int [\mathrm{nM}]^2\frac{dS}{R^2}; \quad h = \frac{1}{\pi}\int [\mathrm{nN}]^2\frac{dS}{R^2}, \qquad (2.18)$$

$$\mathrm{M} = [\mathfrak{f}\rho]; \quad \mathrm{N} = [[\mathfrak{f}\rho]\,\rho].$$

Let us find the fluctuating field in the focus of a paraboloidal mirror. The origin of the coordinate system is placed at the focus in such a way that its axis coincides with the axis of revolution (Fig. 6). It is easy to see that for a paraboloid

$$dS/R^2 = 2\sin\frac{\vartheta}{2}\,d\vartheta\,d\varphi, \qquad (2.19)$$

$$n_x = \sin\frac{\vartheta}{2}\cos\varphi;$$

$$n_y = \sin\frac{\vartheta}{2}\sin\varphi; \quad n_z = \cos\frac{\vartheta}{2}.$$

Forming the quantities n × M and n × N of interest to us, and performing straightforward trigonometric transformations, we obtain, after carrying out the integration in Eq. (2.18):

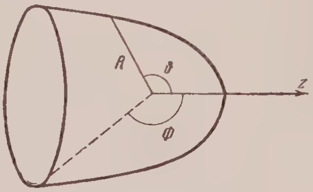

FIG. 6

$$e_x = 4\left[\left(1 - \cos\frac{\Phi}{2}\right) - \left(1 - \cos^3\frac{\Phi}{2}\right) + \frac{4}{5}\left(1 - \cos^5\frac{\Phi}{2}\right)\right]; \qquad (2.20)$$

$$e_z = 32\left[\frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right) - \frac{1}{5}\left(1 - \cos^5\frac{\Phi}{2}\right)\right];$$

$$h_x = 4\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right) - \frac{4}{5}\left(1 - \cos^5\frac{\Phi}{2}\right) + \frac{4}{7}\left(1 - \cos^7\frac{\Phi}{2}\right)\right],$$

$$h_z = 32\left[\frac{1}{5}\left(1 - \cos^5\frac{\Phi}{2}\right) - \frac{1}{7}\left(1 - \cos^7\frac{\Phi}{2}\right)\right],$$

where Φ is the angular opening of the mirror. In particular, for an infinite paraboloid ($\Phi = \pi$)

$$e_x = 16/5 = 3.20; \quad e_z = 64/15 = 4.26;$$

$$h_x = 464/105 = 4.42; \quad h_z = 64/35 = 1.83.$$

Curves calculated on the basis of Eq. (2.20) are given in Fig. (7).

Furthermore, from Eq. (2.20)

$$2e_x + e_z = 2h_x + h_z$$
$$= 8\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right)\right],$$

so that at the focus

$$w_\omega^{\text{el}} = w_\omega^{\text{mag}} = \frac{1}{2}\, w_\omega$$
$$= 2\eta w_{\text{eq.}}\left[\left(1 - \cos\frac{\Phi}{2}\right) + \frac{1}{3}\left(1 - \cos^3\frac{\Phi}{2}\right)\right].$$

In particular, for an infinite paraboloid $w_\omega = (16/3)\eta w_{\text{eq.}}$

As a second example we consider the fluctuating field at the center of a spherical mirror with an angular opening $\Phi \leq \pi/2$ (Fig. 8). In this case, $dS/R^2 = \sin\vartheta d\vartheta d\varphi$. After simple calculations we obtain

$$e_x = h_x = (1 - \cos\Phi) + {}^1\!/_3\,(1 - \cos^3\Phi), \tag{2.21}$$
$$e_z = h_z = 2\,[(1 - \cos\Phi) - {}^1\!/_3\,(1 - \cos^3\Phi)].$$

In particular, for a hemisphere $e_x = e_z = h_x = h_z = 4/3$. Curves based on Eq. (2.21) are given in Fig. (9).

In the case of a spherical mirror, again,

$$w_\omega^{\text{el}} = w_\omega^{\text{mag}} = {}^1\!/_2\, w_\omega,$$

$$a\ \ w_\omega = 2\eta w_{\text{eq.}}(1 - \cos\Phi) = 4\eta w_{\text{eq.}}\Omega/4\pi,$$

where $\Omega = 2\pi(1 - \cos\Phi)$ is the solid angle of the mirror.

For parabolic and spherical mirrors with a small angular opening ($\Phi \ll 1$) Eqs. (2.20) and (2.21) evidently give the same expressions

$$e_x = h_x = \Phi^2; \quad e_z = h_z = {}^1\!/_2\Phi^4.$$

3. Fluctuating Surface Charges

As is well known, in the solution of high-frequency electrodynamical problems in regions containing good conductors with a strongly expressed skin effect, we may limit ourselves to the consideration of the electromagnetic field only exterior to these conductors. It is only necessary to require that on the surface of good conductors, the external field vectors satisfy the approximate Leontovich boundary conditions[9]. Finding the thermal radiation of such bodies may also be treated (see Ref. 2) as an external electrodynamic boundary value problem: the sources of the fluctuating field are the random surface extraneous fields entering into the Leontovich boundary conditions, introduced into the theory as the equivalent of random volume fields. The external radiation of the body is due to the volume extraneous fields distributed in the skin-layer. Hence, the replacement of volume sources by surface sources corresponds formally to the transition to the limit $\delta \to 0$. Therefore, within the limits of such a "surface" treatment of the fluctuating fields using the inhomogeneous Leontovich conditions, it is natural to introduce the random surface charge densities

$$\sigma = (1/4\pi)\,(E_n)_{\text{sur}},$$

which are characterized by the correlation function

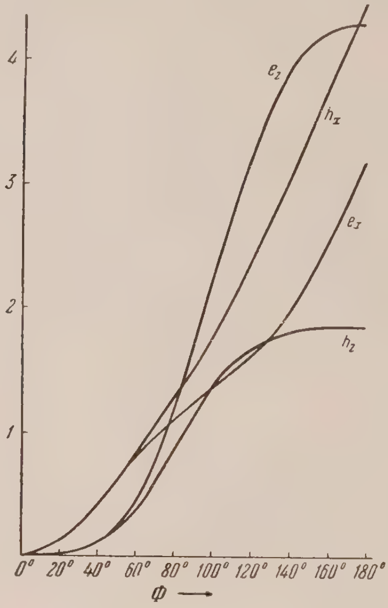

FIG. 7

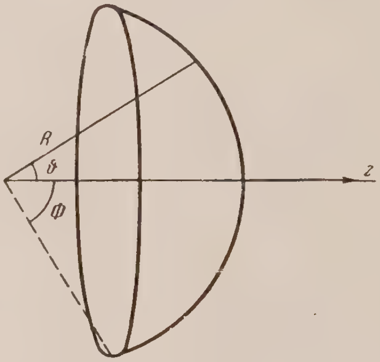

FIG. 8

$$\overline{\sigma_a \sigma_b^*} = (1/16\pi^2)\, (\overline{E_{an} E_{bn}^*})_{\text{sur}}, \tag{2.22}$$

where a and b are points on the surface of the conductor.

Here, we will find this correlation function for distances small in comparison with all dimensions of the body and with the wavelength. Then, clearly, the surface of the body may be replaced by a plane, Eq. (2.22) may be written in the form

$$\overline{\sigma_a \sigma_b^*} = (1/16\pi^2)\, (\overline{E_{az} E_{bz}^*})_{z=0} \tag{2.23}$$

and the right hand side of (2.23) may be calculated by a formula of the type (2.15)

$$\overline{E_{az} E_{bz}^*} = \frac{c\Theta\eta}{\pi^2} \int [\mathbf{n}\mathbf{H}_{1a}]\, [\mathbf{n}\mathbf{H}_{1b}^*]\, dS. \tag{2.24}$$

Furthermore, in the expressions (2.6) for $\mathbf{H}_{1a,b}$ it is possible to leave only quasi-stationary terms

$$\mathbf{H}_{1a} = [\mathbf{n}\rho_a]/cR_a^2; \quad \mathbf{H}_{1b} = [\mathbf{n}\rho_b]/cR_b^2,$$

where we have taken account of the fact that in our case $\mathbf{f}_a = \mathbf{f}_b = \mathbf{n}$.

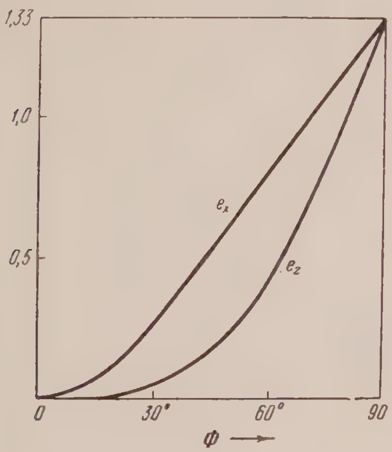

FIG. 9

We will first calculate the right hand side of Eq. (2.24) with z not equal to zero. Let D be the distance between the points a and b at a distance z from the plane; r_a and r_b are the projections of the vectors $\mathbf{R}_a$ and $\mathbf{R}_b$ on the plane, ψ is the angle between these projections (Fig. 10), r and φ are polar coordinates on the plane, the origin of which is midway between the projections of the points a and b. Then $|[\mathbf{n} \times \rho]| = \sin\vartheta = r/R$, so that

$$\overline{E_{az} E_{bz}^*} = \frac{\Theta\eta}{\pi^2 c} \int \frac{r_a r_b \cos\psi}{R_a^3 R_b^3}\, dS. \tag{2.25}$$

Using the theorem concerning the square of a side of a scalene triangle, we easily find

$$r_a r_b \cos\psi = r^2 - D^2/4;$$

$$R_a R_b = [(r^2 + {}^1\!/_4 D^2 + z^2)^2 - r^2 D^2 \cos^2\varphi]^{1/2},$$

and the calculation of the integral in (2.25) gives

$$\overline{E_{az} E_{bz}^*} = (\Theta\eta/2\pi c)\, z\, (z^2 + D^2/4)^{-3/2}. \tag{2.26}$$

But

$$\lim_{z \to 0} \frac{z}{(z^2 + D^2/4)^{3/2}} = 8\pi\delta\, (\mathbf{r}_a - \mathbf{r}_b),$$

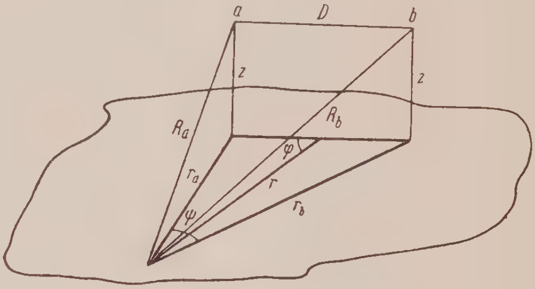

FIG. 10

where $\delta(\mathbf{r}_a - \mathbf{r}_b)$ is the two-dimensional δ-function. Now substituting the limiting expression (2.26) into (2.23), we obtain the correlation function of interest to us

$$\overline{\sigma_a \sigma_b^*} = (\Theta\eta/4\pi^2 c)\, \delta\, (\mathbf{r}_a - \mathbf{r}_b). \tag{2.27}$$

The fluctuating charge relating to the area of the surface S is $q = \int_S \sigma\, dS$. On the basis of Eq. (2.27), its mean square is

$$\overline{|q|^2} = \int_S \cdot \int_S \overline{\sigma_a \sigma_b^*}\, dS_a dS_b = \frac{\Theta\eta}{4\pi^2 c}\, S$$

or, transforming to the spectral densities of the decomposition according to positive frequencies (the remaining random quantities in this paragraph were spectral densities of the interval $-\infty < \omega < +\infty$), we have

$$q_\omega^2 = (\Theta\eta/2\pi^2 c)\, S.$$

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The Scattering of π^+-Mesons on Hydrogen. I. Angular Distributions at Energies of 176, 200, 240, 270 and 307 MEV.

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The angular distribution of π^+-mesons with energies of 176, 200, 240, 270 and 307 mev scattered from liquid hydrogen has been studied with the help of scintillation counters. The total interaction cross section of π^+-mesons and hydrogen has also been measured by the method of attenuation of the beam. At all energies, except 307 mev, the data can be well described by a function of the form $d\sigma/d\Omega = a + b\cos\theta + c\cos^2\theta$.

INTRODUCTION

At this time there are comparatively little accurate data on the angular distribution of π^+-mesons elastically scattered from hydrogen. Most of the published results pertain to energies less than 200 mev.[1-5] This situation can be explained by the difficulty involved in obtaining sufficiently intense and monoenergetic beams of positive π-mesons of high energy from circular accelerators. The measurement carried out to an energy of 400 mev with a diffusion chamber[6] is of little accuracy because of the small number of observed interactions. The scattering of 310 mev π^+-mesons from hydrogen has been studied recently by Grigor'ev and Mitin[7] with photographic plates.

The possibility of obtaining intense beams of positive π-mesons of high energy at the Institute for Nuclear Problems of the Academy of Sciences of the USSR appeared only after the extraction from the synchrocyclotron chamber of a beam of protons with an energy of approximately 660 mev and a density $\sim 10^9$ / cm^2 / sec.[8]

In the present communication the results of work[9] on the study of the scattering of π^+-mesons from hydrogen at meson energies of 176, 200, 240, 270 and 307 mev are presented in detail. The measurements were made with the help of scintillation counters.

BEAMS OF π^+-MESONS

Beams of π^+-mesons were obtained by the bombardment of a polyethylene target with a proton beam extracted from the synchrocyclotron. Mesons, formed in the reaction $p + p \rightarrow \pi^+ + d$ at an angle of 9° with the incident proton beam, were magnetically deflected to an angle of 20° and, passing through an iron collimator four meters thick with an internal diameter of 5 cm, struck the detecting equipment placed behind the principal shielding wall of the synchrocyclotron (Fig. 1).

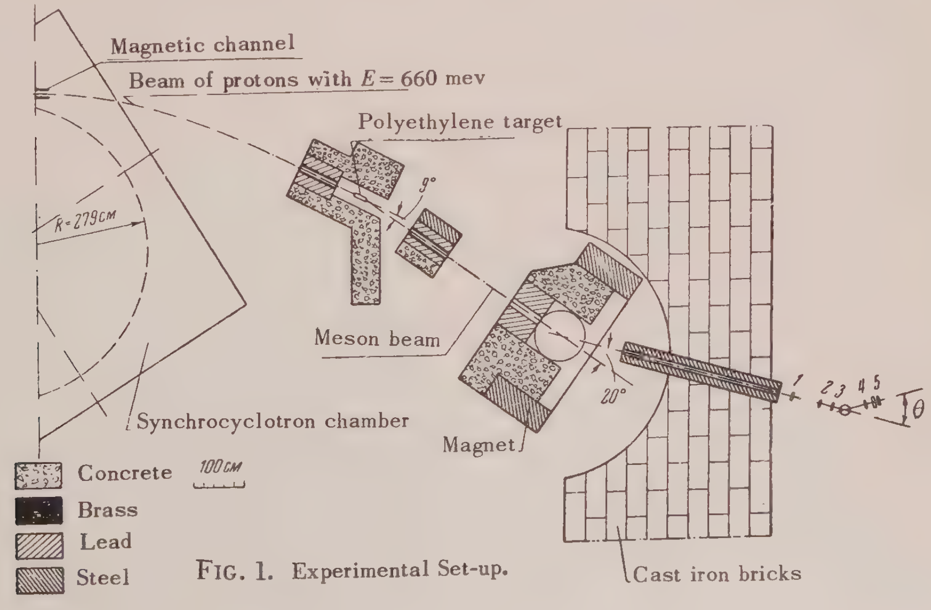

FIG. 1. Experimental Set-up.

For each desired meson energy, determined by the magnitude of the magnetic field of the deflecting magnet, the thickness of the polyethylene target was experimentally chosen in such a way as to obtain the maximum meson flux. The optimal thicknesses of the polyethylene target for energies of 176, 200, 240, 270 and 307 mev were, respectively, approximately 67, 53, 35, 19 and 4.6 gm/cm ^{2}. In these cases the obtained fluxes of mesons were approximately equal to 20, 40, 60, 60 and 40 cm^{-2} sec^{-1}.

It is interesting to note that despite the appreciable thicknesses of the targets used in these experiments there is a relatively narrow peak from the reaction $p + p \rightarrow \pi^+ + d$ (Fig. 2) in the relation between the intensity of the meson beam, Φ , and the current in the deflecting magnet, I_M . This can be explained by the fact that mesons created in the forward part of the target and stopping in the target have approximately the same energy as mesons created by a slowed-down beam of protons in the "backward" part of the target, at their emergence from the target. The accurate determination of the energy in each case was carried out by measuring the range of mesons in copper. A typical absorption curve for 307 mev mesons is shown on Fig. 3. The range found from the curve was then corrected, as before,[10] for the calculated multiple Coulomb scattering of mesons in lead. For all energies this correction was approximately 2.5 % of the measured range and the energy essentially does not change due to the correction. To the "corrected" range was added 0.4 gm/cm^2 of copper, which is equivalent to the wall thickness of the photomultiplier container and the thickness of the plexiglass wall of the last scintillator. The final values of the energies, presented in Table 1, take into account the slowing down of mesons in the walls of the Dewar and in the hydrogen and correspond to the energy of the mesons in the center of the target. The uncertainty in the energy is due to the fundamental inaccuracy in the range determination, the initial inhomogeneity of the beam and also the inaccuracy in the calculation of the range-energy curve.

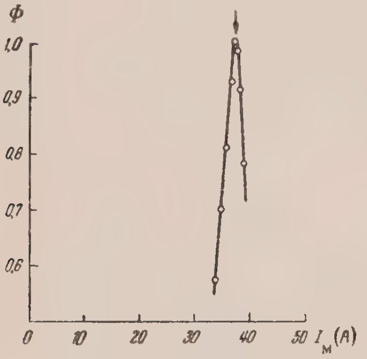

FIG. 2. Dependence of the intensity of the meson beam on the current in the deflecting magnet I_M for a polyethylene target thickness of 35 gm/cm^2. E_π =240 mev.

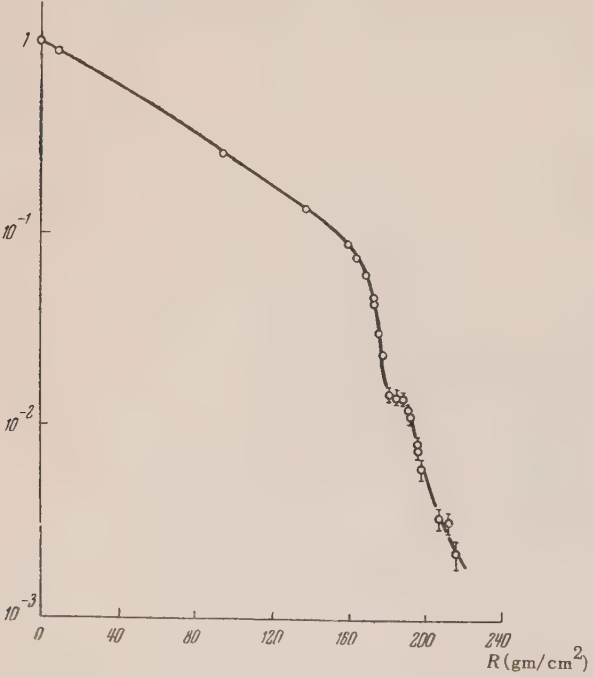

FIG. 3. Typical absorption curve (E_π=307 mev).

From the absorption curves and from known data on the nuclear interaction of π-mesons with copper, the admixture of μ-mesons, which is also shown in Table 1, was determined.

EXPERIMENTAL ARRANGEMENT

A. ANGULAR DISTRIBUTION

The geometry of the experiment for the measurement of angular distributions is given in Fig. 4. In the experiments a Cerenkov counter, *1*, was used as well as liquid scintillation counters *2* and *3*, set in coincidence for the registration of the number of π^+-mesons hitting the hydrogen target (monitor). The Cerenkov detector was chosen in view of the fact that the meson beam contains an appreciable number of slow protons. The scattering of mesons from hydrogen was detected by liquid scintillation counters *4* and *5*, set in coincidence with counters *1* and *3*. The block diagram of the electronic equipment is shown on Fig. 5 The resolving time of the three-fold and four-fold coincidence schemes was equal to 1.5×10^{-8} sec. The stability of operation of the electronic equipment was systematically checked by the measurement of the ratio N_4 / N_3 , the number of four-fold

TABLE 1

Corrected range of π^+-mesons in copper, in gm/cm^2	Energy of the π^+-mesons in the center of the hydrogenous scatterer, in mev	Uncertainty in the energy of the π^+-mesons interacting with hydrogen, in mev	Percent admixture of μ^+-mesons
90.4	176	±4	4.7±1.5
106.6	200	±5	4.0±1.0
134.1	240	±6	3.7±1.0
155.2	270	±6	3.4±1.0
180.4	307	±6	2.6±1.0

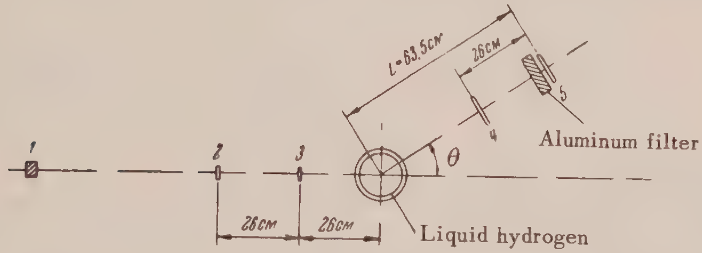

FIG. 4. Arrangement of the equipment used during the angular distribution measurements. *1*—Cerenkov detector; *2,3,4,5*—scintillation counters. For $\theta = 20°$, $L = 73.5$ cm.

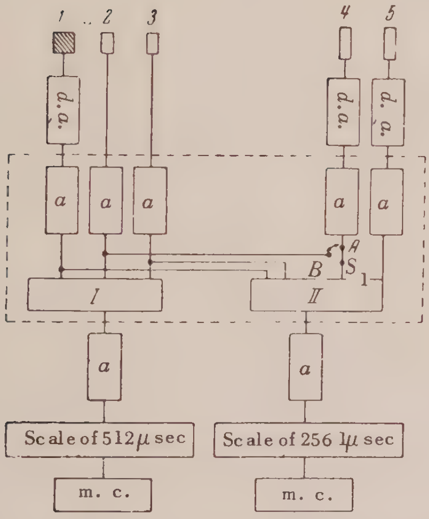

FIG. 5. Block diagram of the electronic equipment. S_1 = switch. Differential cross sections were measured with the switch at point *A* (four-fold coincidences *1,3,4,5*); total cross sections with the switch at point *B* (four-fold coincidences *1,2,3,5*). *A*—amplifier, *d.a.*—distributing amplifier; *I*—3-fold coincidence scheme ~ 1.5 × 10^{-8} sec; *II*—4-fold coincidence scheme ~ 1.5 × 10^{-8} sec; *1*—Cerenkov counter; *2,3,4,5* — scintillation counters; *m.c.*—mechanical counter.

coincidences (1,3,4,5) and the number of three-fold coincidences (1,2,3) for which the counters *4* and *5* were set in one line with the counters *1,2* and *3*.

Counter *1*——the Cerenkov radiation detector——was built of plexiglass having dimensions of 5 × 5 cm and a thickness of 3.6 cm and was connected by a plexiglass light pipe with a photomultiplier tube FEU–19. The scintillation counters *2* and *3* had dimensions 5 × 5 × 0.6 cm and consisted of a thin-walled plexiglass vessel filled with a solution of terphenyl in phenylcyclohexane. Counters *4* and *5* had a diameter and thickness of scintillating material (a solution of terphenyl in phenylcyclohexane) of 10 and 0.8 cm, respectively. The scintillators were connected by light pipes to photomultiplier tubes FEU–19.

A glass-walled Dewar with an internal diameter of about 14 cm and a wall thickness of 1 gm/cm^2 was used for the hydrogen target. For a Dewar capacity of 3 liters and an average evaporation of the hydrogen of about 0.25 *l* / hr, uninterrupted work without additional filling of liquid hydrogen was assured for a period of 8 hours. During the experiments, the Dewar was pumped out for background determinations (without hydrogen).

It was assumed in determining the amount of hydrogen that the density of liquid hydrogen was 0.0708 gm/cm^3. Taking into account the fact that the evaporation speed of the hydrogen was relatively small, the influence of bubbles on the density was neglected. The average amount of hydrogen in the path of the meson beam determined under the assumptions mentioned above was 0.984 gm/ cm^2 with an uncertainty of about 1%. In order not to register charged particles from stars formed in the walls of the Dewar or recoil protons from (π^+,p) scattering, an aluminum filter was placed between scintillators *4* and *5*.

The angular distributions were obtained from the ratio N_4 / N_3 after calculation of the detection efficiency of mesons under various angles and a number of other corrections which will be discussed in the next section.

B. TOTAL CROSS SECTION

The total interaction cross section of π^+ -mesons with hydrogen was measured by the absorption of the beam of mesons passing through the hydrogen scatterer in the geometry shown in Fig. 6.

As in previous work on the determination of total cross sections, the distance between the center of the hydrogen scatterer and the last scintillator was equal to 35 cm, which corresponds to an average detection angle for the last detector of about 8°.

The beam of mesons falling on the hydrogen target was defined by the system of three counters *1,2* and *3* set in coincidence. Having passed through the scatterer the beam was registered through a four-fold coincidence (1,2,3,5). The block diagram of the electronic equipment for these experiments is also shown in Fig. 5.

The scintillators *2* and *3* used in these experiments had a diameter of 3 cm and a thickness of 0.4 cm and consisted of a thin-walled plexiglass container filled with a solution of terphenyl in phenylcyclohexane. The scintillator *5* had a diameter of 10 cm as before. The same hydrogen target was used as in the measurments of the angular distributions.

THE METHOD OF THE MEASUREMENT OF ANGULAR DISTRIBUTIONS

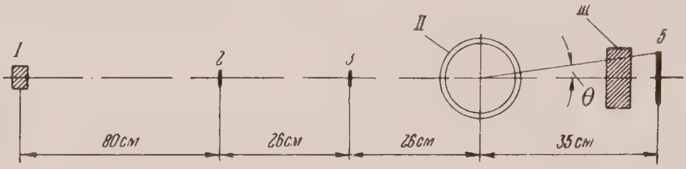

FIG. 6. Set-up of the equipment during the total cross section measurements. *I*—Cerenkov detector; *2,3,5*— Scintillation counters; *II*—Dewar with liquid hydrogen, *III* — CH_2 filter (5cm thick).

The differential cross section for the elastic scattering of π^+ -mesons on hydrogen in the laboratory system was determined from the formula

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{lab}} = \left(\frac{N_4}{KN_3}\right)\left(\frac{1}{N_H\epsilon\Omega}\right),$$

where N_4 is the number of registered four-fold coincidences of π^+ -mesons scattered from hydrogen at a given angle θ; N_3 is the number of three-fold coincidences; K is a factor characteristic of the beam; N_H is the number of hydrogen atoms per cm^2; ϵ is the detection efficiency for mesons at angle θ ; Ω is the effective solid angle. A discussion is presented below of experiments carried out to find the parameters entering in the formula shown above. The numerical values of the parameters will be shown in tables in which are presented the results of the measurements.

A. DETERMINATION OF THE FLUX OF π^+ – MESONS

Because of the fact that in the scattering experiments a "thick" hydrogen target was used, it was necessary to take into account the absorption of the mesons not only in scintillator *3* and in the front walls of the Dewar but also in the hydrogen scatterer itself. In the determination of the number of mesons hitting the hydrogen target, counting errors in the counting scheme were taken into account; these occurred for relatively high intensities used in meson flux experiments, particularly for energies of 240 and 270 mev. Finally, it was necessary to consider the admixture in the beam of μ-mesons which do not participate in nuclear interaction with hydrogen.

Taking into account the facts enumerated above, the average number of π^+-mesons, $N_\pi = KN_3$,which could participate in the scattering were determined in the following way:

$$N_\pi = KN_3 = N_3\eta\alpha_\pi\{1 - {}^1/_2 \exp[-\sigma_t N_{\mathrm{H}}] - \beta\}.$$

Here η is a coefficient which takes into account errors in counting during the registration of the number of three-fold coincidences; α_π is the fraction of π-mesons in the beam; σ_t is the total interaction cross section of π^+-mesons with hydrogen; β is the fraction of the beam absorbed in scintillator *3* and in the front walls of the Dewar.

B. DETECTION EFFICIENCY OF THE MESONS

The efficiency of detection by four-fold coincidences (1,3,4,5) of mesons scattered in various directions is determined by two factors: 1) the probability of the registration of mesons crossing the sensitive volume of scintillation counters *4* and *5*, and 2), the probability of the absorption of mesons scattered at a given angle into scintillator *4* and the aluminum filter. Both the absorption of scattered mesons and the probability of their detection were determined in separate experiments. The first factor was determined in experiments carried out with the arrangement of the scintillation counters shown in Fig. 7.

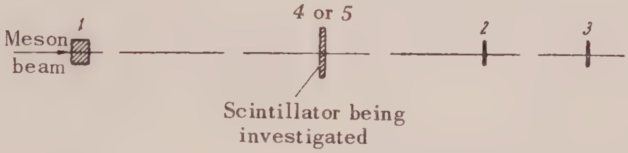

FIG. 7. Set-up of the experiment to determine the probability of the detection of mesons in scintillators *4* and *5*.

It is found that the probability of detecting mesons in scintillators *4* and *5* $\epsilon_4 = N_4(1,2,3,4)/N_3(1,2,3)$ and $\epsilon_5 = N_4(1,2,3,5)/N_3(1,2,3)$ were equal to unity with an accuracy of better than one percent.

The plateau of the system of the four counters *1,3,4* and *5* set up in a straight beam had a slope of the order of 1% per 200-300 volts.

To measure the absorption of scattered mesons in "equivalent filters" consisting of scintillator *4* and aluminum filters, counters *1,2,3* and *5* were placed in a single line. The attenuation of meson beams in these equivalent filters was determined for a number of energies, using a variation of the electronic scheme (see Fig. 5) intended for the measurement of total cross sections. The geometry of the experiment is shown in Fig. 8. In these experiments the energy of the meson beam was set by the magnitude of the current in the deflecting magnet. The control measurement carried out by measuring the range in copper showed that the divergence between the energy values determined in both ways was less than ±2 mev.

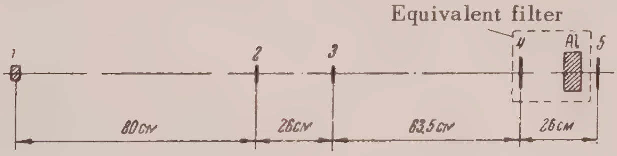

FIG. 8. Arrangment by which the attenuation of the meson beam in the filter was measured.

From the combined results, the measurements of

$$\epsilon_\Phi = \frac{N_4\ \text{(with filter)}}{N_4\ \text{(without filter)}}$$

as a function of energy, taken for various thicknesses of the aluminum filter, show that for energies near 200 mev, $\epsilon_\Phi(E)$ has a minimum, which can be understood from the character of the energy dependence of the total interaction cross section of π-mesons with complex nuclei.[12]

From the curves characterizing the absorption of mesons as a function of energy for various thicknesses of aluminum filter (Fig. 9), the detection efficiency for scattered mesons of a given energy was directly found, since the probability of detecting mesons traversing scintillators *4* and *5* is equal to unity. As a small correction to the detection efficiency the absorption of scattered mesons in the walls of the hydrogen target was taken into account. In such a way the detection efficiency of scattered mesons was determined to be

$$\varepsilon = \varepsilon_{\Phi}(1 - \varkappa),$$

where $\varkappa$ is the fraction of scattered mesons absorbed in the Dewar walls.

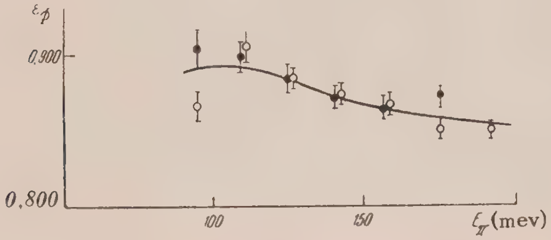

FIG. 9. Detection efficiency of mesons (for an aluminum filter thickness of 3 cm) as a function of energy.

C. DETERMINATION OF THE EFFECTIVE SOLID ANGLE

The most substantial of the geometrical corrections depends on the fact that the dimensions of the scintillator *4* were less than the diameter of the hydrogen target. This leads to the fact that, depending on the angle, part of the hydrogen scatterer is in a half shadow with respect to the counter telescope *4-5*; Fig. 10 shows the dependence on angle θ of the ratio Ω_{av}/Ω_c where Ω_c is the solid angle defined by scintillator *5* as seen from the center of the hydrogen target, and Ω_{av} is the solid angle determined by the scintillation counter telescope *4-5*, averaged with respect to the volume of the scatterer.

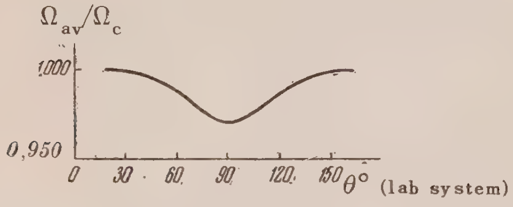

FIG. 10. Angular dependence of the quantity Ω_{av}/Ω_c

In order to obtain the effective solid angle of detection of scattered mesons, Ω, it is nesessary to introduce a further correction to the quantity Ω_{av}, taking into account the possibility of detecting scattered mesons in the light pipe of scintillation counter 5. The relatively large sensitivity of the plexiglass light pipe to high energy particles results from Cerenkov radiation and from "feeble" scintillations in the plexiglass.

The increase of the average solid angle $(\Omega - \Omega_{av}) \times \Omega_{av}$ was determined for mesons of known energy scattered through two angles. The relative dependence of the sensitivity of the light pipe on energy was determined in the direct meson beam for various values of the current in the deflecting magnet. This gave the possibility of obtaining the magnitude of the effective solid angle, Ω, essential to the determination of the differential scattering cross section for mesons of various energies brought up in this work. The dependence of the increase of the average solid angle on the meson energy is shown in Fig. 11. In the determination of the effective solid angle a correction was also introduced due to the fact that the scintillation counter telescope has a finite angular resolution.

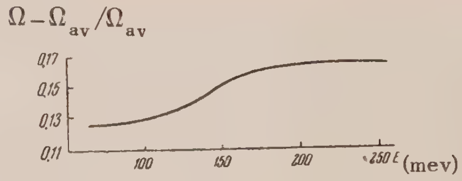

FIG. 11. Energy dependence of the quantity $(\Omega - \Omega_{av})/\Omega_{av}$.

Processes of multiple nuclear scattering of mesons in hydrogen were not taken into account. Because of their small magnitude, corrections due to chance coincidence and $\pi-\mu$ decay of scattered mesons were not taken into account.

THE METHOD OF MEASUREMENT OF TOTAL CROSS SECTIONS

Total interaction cross sections of π^+-mesons with hydrogen, measured by the method of attenuation of the beam traversing the scatterer, were determined from the formula

$$\sigma_t(\pi^+, p) = \frac{1}{N_{\mathrm{H}}} \ln \left\{ \frac{N_4^{(\mathrm{no\,H})}\eta^{(\mathrm{no\,H})} - N_4^{(0)}\eta^{(0)}\alpha_\mu}{N_4^{(\mathrm{H})}\eta^{(\mathrm{H})} - \Lambda_4^{(0)}\eta^{(0)}\alpha_\mu} \right\} + \int_0^{\vartheta_1} \left\{ \left(\frac{d\sigma_\pi}{d\Omega}\right)_{\mathrm{lab}} + \left(\frac{d\sigma_p}{d\Omega}\right)_{\mathrm{lab}} \right\} d\Omega.$$

The following terms are used here: $N_4^{(0)}$, $N_4^{(\mathrm{noH})}$ and $N_4^{(\mathrm{H})}$ are respective to the number (having normalized to unity the reading of the monitor) of four-fold coincidence counts corresponding to the hydrogen free Dewar ($N_4^{(0)}$), pumped out Dewar

($N_4^{(\text{noH})}$) and Dewar filled with liquid hydrogen ($N_4^{(\text{H})}$); $\eta^{(0)}$, $\eta^{(\text{noH})}$ and $\eta^{(\text{H})}$ are factors which take into account counting errors in the detecting apparatus, α_μ is the fraction of μ-mesons in the beam; $(d\sigma_\pi / d\Omega)_{\text{lab}}$ is the angular distribution of π-mesons in the laboratory system of coordinates; $(d\sigma_p / d\Omega)_{\text{lab}}$ is the angular distribution of recoil protons in the laboratory system of coordinates; $\theta_1 = 8°$.

Chance coincidences, determined in a separate experiment, were negligibly small (of the order of 0.1%) and were not taken into account.

THE RESULTS OF THE MEASUREMENTS

A. DIFFERENTIAL CROSS SECTIONS

The results of the measurements are presented in Tables 2–6. Besides the final values of the differential cross sections in the laboratory system $(d\sigma/d\Omega)_{\text{lab}}$ and in the center-of-mass system $(d\sigma/d\Omega)$, previously determined numerical values of the parameters K, Ω and ϵ are also given. One can obtain an idea of the magnitude of the background at various angles in the absence of hydrogen from the fourth column of Tables 2–6 where the ratio N_4 (Dewar)/N_4 (Dewar + H), the number of four-fold coincidences normalized to unity on the monitor, is given. In the fifth column of these tables is shown the calculation [N_4 (Dewar + H) − N_4 (Dewar)] of four-fold coincidences, dependent only upon hydrogen and normalized to 10^6 counts of the monitor, that is three-fold coincidences, N_3.

Errors in the final cross section results are standard deviations. They include errors in the determination of corrections and they have only a statistical nature in the final calculation.

In regard to other supplemenatry errors of unknown origin, it is seen that it is unlikely that they are of appreciable magnitude because of the closeness of the value of the total cross section $\sigma_t(\pi^+,p)$, determined independently by the attenuation of the meson beam, and the value of

$$\sigma_t = \int (d\sigma/d\Omega)\, d\Omega,$$

obtained by integrating the differential cross sections. However the presence of such errors indicate the magnitude of the deviation of the experimental points from the "best" curve computed by the method of least squares.

At first, after considering the obtained results, a very much simplified phase analysis was made in the belief that only S- and P-waves participate in the scattering, that is that the angular distributions have the form

$$d\sigma/d\Omega = a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta).$$

The corresponding coefficients for various energies, found by the method of least squares, are presented in Table 7. For ease in comparing the results of this work with results of measurements of angular distributions at other energies, usually shown in the form $d\sigma/d\Omega = a + b\cos\theta + \cos^2\theta$, the corresponding coefficients are given in Table 8.

In the last column of Table 7 the values of

$$M = \sum_i \left\{\frac{f(\theta_i) - \sigma(\theta_i)}{\Delta\sigma(\theta_i)}\right\}^2,$$

are given where $\sigma(\theta_i)$ are the measured differential cross sections at an angle θ_i with an experimental error $\Delta\sigma(\theta_i)$; $f(\theta_i)$ is the value of the function $a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta)$ at an angle θ_i found by the method of least squares, In the next to the last column the expected value M' of this quantity is given; it is equal to the difference between the number of experimental points and the numbers of free parameters.[13] As can be seen from the Table, for energies of 176, 200 and 240 mev, the value of M is close to the expected value. For an energy of 307 mev, M appreciably exceeds M'. This is illustrated in Figs. 12-16 where, together with the experimental cross sections, distributions of the type

$$d\sigma/d\Omega = a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta)$$

are shown.

Hence, for a meson energy of 307 mev there is definite sense in approximating the experimental data with a function having a larger number of free parameters, that is, a function of the type

$$d\sigma/d\Omega = a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta) + a_3P_3(\cos\theta) + a_4P_4(\cos\theta).$$

This may mean that for energies of ~ 300 mev meson-nucleon scattering processes are already difficult to describe by means of S- and P-waves alone. The importance of the D-wave contribution will be discussed later. Table 9 gives the angular distribution coefficients, found by the method of least

TABLE 2. Angular distribution of π^+ – mesons scattered from hydrogen, $E_\pi = 176 \pm 4$ mev.

Angle θ (lab system), in degrees	Distance from the center of the target to the last scintillator in cm	Thickness of the aluminum filter in cm	Approximate magnitude of the background $\frac{N_4\text{ (Dewar)}}{N_4\text{(Dewar+ H)}}$	"Clean" effect from hydrogen (for 10^6 counts of the monitor	K, beam characteristic factor	Ω, effective solid angle in the lab system, in steradians	ϵ, detection efficiency of the scattered mesons	$(d\sigma/d\Omega)_{\text{lab}}$, differential cross section in the lab system, in 10^{-27} cm^2 sterad	Angle θ in the center-of-mass, in degrees	$d\sigma/d\Omega$ differential cross section in the center-of-mass system, in 10-27 cm^2/sterad
30	63.5	3.8	0.4	308±30	0.915	2.20×10^{-2}	0.835	31.1±3.1	38.5	20.0±2.0
45	63.5	3.0	0.2	221±20	0.915	2.16×10^{-2}	0.858	22.1±2.1	56.8	15.7±1.5
60	63.5	3.0	0.14	129±14	0.915	2.17×10^{-2}	0.867	12.7±1.4	74.2	10.2±1.1
75	63.5	3.0	0.17	96±11	0.915	2.13×10^{-2}	0.879	9.6±1.1	90.4	8.8±1.0
90	63.5	0.8	0.17	105±13	0.915	2.10×10^{-2}	0.912	10.2±1.3	105.6	10.8±1.4
105	63.5	0.8	0.13	128±14	0.915	2.10×10^{-2}	0.904	12.5±1.4	119.5	15.3±1.7
120	63.5	0.8	0.16	142±15	0.915	2.11×10^{-2}	0.900	13.9±1.5	132.9	19.3±2.1
135	63.5	0.8	0.22	146±15	0.915	2.13×10^{-2}	0.898	14.2±1.5	145.3	21.8±2.3
149	63.5	0.8	0.10	175±16	0.915	2.15×10^{-2}	0.898	16.8±1.5	157.2	27.9±2.6

TABLE 3. Angular distribution of π^+ – mesons scattered from hydrogen, $E_\pi = 200 \pm 5$ mev.

Angle θ (lab system), in degrees	Distance from the center of the target to the last scintillator in cm	Thickness of the aluminum filter, in cm	Approximate magnitude of the background $\frac{N_4\text{ (Dewar)}}{N_4\text{ (Dewar + H)}}$	"Clean" effect from hydrogen (for 10^6 counts of the monitor)	K, beam characteristic factor	Ω, effective solid angle in the lab system, in steradians	ϵ, detection efficiency of the scattered mesons	$(d\sigma/d\Omega)_{\text{lab}}$, differential cross section in the lab system, in 10^{-27} cm^2/sterad	Angle θ in the center-of-mass, in degrees	$d\sigma/d\Omega$ differential cross section in the center-of-mass system, in 10-27 cm^2/sterad
20	73.5	6.8	0.6	249±21	0.930	1.68×10^{-2}	0.734	36.9±3.2	26.3	22.1±1.9
35	63.5	6.8	0.3	242±16	0.931	2.23×10^{-2}	0.733	27.0±1.9	45.3	17.5±1.2
50	63.5	3.0	0.2	199±12	0.960	2.22×10^{-2}	0.850	18.7±1.2	63.5	13.6±0.9
80	63.5	3.0	0.2	77.6±6.4	0.949	2.18×10^{-2}	0.870	7.38±0.75	96.5	7.13±0.73
95	63.5	3.0	0.1	93±11	0.930	2.13×10^{-2}	0.883	9.0±1.1	111.2	10.2±1.2
110	63.5	3.0	0.1	117.1±8.3	0.964	2.14×10^{-2}	0.893	10.8±0.8	125.8	14.1±1.1
150	63.5	3.0	0.2	141±10	0.940	2.18×10^{-2}	0.881	13.3±1.0	157.4	22.8±1.7

TABLE 4. Angular distribution of π^+–mesons scattered from hydrogen, $E_\pi = 240 \pm 6$ mev.

Angle θ (lab system), in degrees	Distance from the center of the target to the last scintillator, in cm	Thickness of the aluminum filter, in cm	Approximate magnitude of the background $\frac{N_4\,(\text{Dewar})}{N_4\,(\text{Dewar}+\text{H})}$	"Clean" effect from hydrogen (for 10^6 counts of the monitor	K, beam characteristic factor	Ω, effective solid angle in the lab system, in steradians	ϵ, detection efficiency of the scattered mesons	$(d\sigma/d\Omega)_{\text{lab}}$, differential cross section in the lab system, in 10^{-27} cm^2/sterad	Angle θ in the center-of-mass, in degrees	$d\sigma/d\Omega$ differential cross section in the center-of-mass system, in 10^{-27} cm^2/sterad
20	73,5	9.0	0,6	267 ± 25	0.999	$1,69 \times 10^{-2}$	0,701	$38.4 \pm 3,7$	26.8	$22.0 \pm 2,1$
35	73,5	6.8	0,2	166 ± 12	0.999	1.69×10^{-2}	0.749	$24.6 \pm 1,6$	46.3	15.4 ± 1.0
35	63.5	6,8	0,3	174 ± 18	0,999	$2,25 \times 10^{-2}$				
50	63,5	3.0	0,3	155 ± 10	1,074	$2,25 \times 10^{-2}$	0,823	$13,3 \pm 1,0$	64.7	$9,47 \pm 0,68$
80	63.5	3,0	0,3	$59,9 \pm 4,0$	1,074	$2,20 \times 10^{-2}$	0.862	$5,00 \pm 0.36$	97.8	$4,92 \pm 0,35$
95	63.5	3,0	0,2	$59,7 \pm 4,2$	1,066	$2,16 \times 10^{-2}$	0.865	$5,10 \pm 0.38$	112,4	$5,86 \pm 0,44$
110	63,5	3.0	0,3	$60,4 \pm 4.2$	1.067	2.15×10^{-2}	0,877	$5,10 \pm 0.38$	126.0	$6,80 \pm 0,51$
150	63,5	3,0	0,2	$97,5 \pm 5,1$	1.038	$2,19 \times 10^{-2}$	0,900	$8,12 \pm 0.47$	158,1	$14,50 \pm 0,84$

TABLE 5. Angular distribution of π^+–mesons scattered from hydrogen. $E_\pi = 270 \pm 6$ mev.

Angle θ lab system), in degrees	Distance from the center of the target to the last scintillator, in cm	Thickness of the Aluminum filter, in cm	Approximate magnitude of the background $\frac{N_4\,(\text{Dewar})}{N_4\,(\text{Dewar}+\text{H})}$	"Clean" effect from hydrogen (for 10^6 counts of the monitor)	K, beam characteristic factor	Ω, effective solid angle in the lab system, in steradians	ϵ, detection efficiency of the scattered mesons	$(d\sigma/d\Omega)_{\text{lab}}$, differential cross section in the lab system, in 10^{-27} cm^2/sterad	Angle θ in the center-of-mass, in degrees	$d\sigma/d\Omega$ differential cross section in the center-of-mass system, in 10^{-27} cm^2/sterad
20	73,5	9.8	0.6	210 ± 16	1,042	$1,69 \times 10^{-2}$	0,689	29.4 ± 2.4	27,5	16.4 ± 1.3
50	63,5	3,8	0,3	$104,6 \pm 6,4$	1,025	$2,26 \times 10^{-2}$	0,827	$9,24 \pm 0,61$	65,7	6.52 ± 0.43
80	63,5	3.0	0,4	$29,3 \pm 2,6$	1,026	$2,23 \times 10^{-2}$	0,848	$2,57 \pm 0.24$	98,8	$2,52 \pm 0.23$
95	63,5	3,0	0,4	$21,9 \pm 3,8$	1,032	$2,15 \times 10^{-2}$	0,859	$1,94 \pm 0,34$	113,3	$2,26 \pm 0,40$
110	63,5	3,0	0,3	$37,1 \pm 2,9$	1,028	$2,16 \times 10^{-2}$	0,871	$3,25 \pm 0,26$	126,8	$4.46 \pm 0,36$
149	63,5	3,0	0,4	$45,2 \pm 3,2$	1.021	$2,19 \times 10^{-2}$	0.880	$3,90 \pm 0.29$	157,7	$7,22 \pm 0,54$

TABLE 6. Angular distribution of π^+-mesons scattered from hydrogen, $E_\pi = 307 \pm 6$ mev.

Angle θ (lab system), in degrees	Distance from the center of the target to the last scintillator, in cm	Thickness of the aluminum filter, in cm	Approximate magnitude of the background N_4 (Dewar) / N_4 (Dewar + H)	"Clean" effect from hydrogen (for 10^6 counts of the monitor	K, beam characteristic factor	Ω, effective solid angle in the lab system, in steradians	ϵ, detection efficiency of the scattered mesons	$(d\sigma/d\Omega)_{\text{lab}}$, differential cross section in the lab system in 10^{-27} cm^2/sterad	Angle θ in the center-of-mass, in degrees	$d\sigma/d\Omega$ differential cross section in the center-of mass system, in 10^{-27} cm^2/sterad.
20	73.5	11.0	0,6	186±15	1.024	$1{,}69 \times 10^{-2}$	0.672	27.2±2.3	27.9	14,5±1,2
35	63.5	6.0	0,3	174±17	0.984	2.25×10^{-2}	0,804	16,6±1,7	47.9	9,86±0,99
50	63.5	5.0	0.3	96.1±5.1	1.021	$2{,}26 \times 10^{-2}$	0,829	8,55±0.48	67.1	5.91±0.33
80	63.5	2.0	0.5	21.1±2.2	1,021	2.25×10^{-2}	0.908	1.72±0.19	100.0	1.55±0,17
95	63.5	3.0	0.4	22.8±4.5	0,992	2.18×10^{-2}	0,850	2.10±0.41	114,5	2.49±0.49
110	63.5	2.0	0.3	29.1±2.6	1,021	$2{,}19 \times 10^{-2}$	0,906	2.44±0,23	127.8	3,42±0.32
130	63.5	3.0	0.3	26.0±4.4	0.996	$2{,}18 \times 10^{-2}$	0.875	2.33±0.40	144.0	3,92±0.67
149	63.5	2.0	0.4	30.1±3.2	1,020	$2{,}19 \times 10^{-2}$	0,915	2,50±0,27	158,2	4.77±0,52

TABLE 7. The scattering of π^+-mesons from hydrogen. The coefficients of the angular distribution.

Energy of the mesons, in mev	$d\sigma/d\Omega = a_0 + a_1 P_1 (\cos\theta) + a_2 P_2 (\cos\theta)$			$\lambda^{-2}\, d\sigma/d\Omega = A_0 + A_1 P_1 (\cos\theta) + A_2 P_2 (\cos\theta)$			Expected value $M' = n - m$	M
	a_0	a_1	a_2	A_0	A_1	A_2		
	10^{-27} cm^2/sterad							
176±4	15,86±0.38	−0,84±0,76	13,5±1,1	1,926±0,047	−0,102±0,093	1,63±0,13	6	1,7
200±5	14.15±0,29	1,04±0,55	12,47±0,82	1,996±0,041	0,147±0.078	1,76±0,11	4	5,7
240±6	10.12±0,29	4,58±0,57	9,95±0.63	1,770±0,050	0,80±0,10	1,74±0,11	4	4,4
270±6	6.47±0,25	4,73±0,50	7,07±0,50	1,300±0,050	0,95±0,10	1,42±0,10	3	5,0
307±6	5.45±0,19	5,12±0,38	6,18±0,40	1,275±0,045	1,199±0,089	1,45±0,09	5	11,3

TABLE 8. The scattering of π^+ – mesons from hydrogen. The coefficients of the angular distribution.

Energy of the mesons, in mev	$d\sigma/d\Omega = a + b\cos\theta + c\cos^2\theta$			$\lambda^{-2}\, d\sigma/d\Omega = A + B\cos\theta + C\cos^2\theta$		
	a	b	c	A	B	C
	10^{-27} cm^2 / sterad^{-1}					
176	9,14±0.42	—0.84±0.76	20.2±1.6	1.11±0.05	—0.10±0.09	2.45±0.19
200	7.92±0.36	1,04±0.55	18.7±1.2	1.12±0.05	0.15±0.08	2,64±0.17
240	5.15±0.22	4.58±0.57	14,93±0.95	0,90±0.03	0.80±0.10	2.61±0.17
270	2.93±0.15	4.73±0,50	10,59±0.75	0.59±0.03	0.95±0.10	2.13±0.15
307	2,36±0.14	5.12±0.38	9.26±0.60	0.55±0.03	1.20±0.09	2.17±0.14

TABLE 9. The scattering of π^+ – mesons from hydrogen (E_π = 307 mev). The angular distribution coefficients.

$d\sigma/d\Omega = a_0 + a_1 P_1(\cos\theta) + a_2 P_2(\cos\theta) + a_3 P_3(\cos\theta) + a_4 P_4(\cos\theta)$					$\lambda^{-2}\, d\sigma/d\Omega = A_0 + A_1 P_1(\cos\theta) + A_2 P_2(\cos\theta) + A_3 P_3(\cos\theta) + A_4 P_4(\cos\theta)$					M'	M
a_0	a_1	a_2	a_3	a_4	A_0	A_1	A_2	A_3	A_4		
10^{-27} cm^2 / sterad^{-1}											
5,49±0.21	5.22±0.49	6.05±0.61	0,43±0.58	—1.34±0.52	1.285±0.049	1.22±0.11	1.42±0.14	0.10±0.14	—0.31±0.1	3	1.4

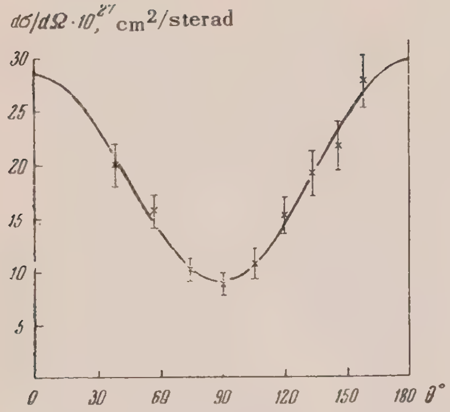

FIG. 12. Angular distribution of 176 mev π^+-mesons scattered from hydrogen. The full curve is for a distribution of the form $d\sigma/d\Omega = [15.9 - 0.8\,P_1(\cos\theta) + 13.5\,P_2(\cos\theta)] \times 10^{-27}$ cm^2/sterad with coefficients derived by the method of least squares.

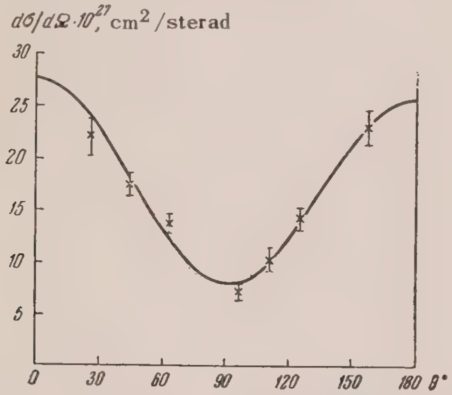

FIG. 13. Angular distributions of 200 mev π^+-mesons scattered from hydrogen. The full curve is for a distribution of the form $d\sigma/d\Omega = [14.2 + 1.0\,P_1(\cos\theta) + 12.5\,P_2(\cos\theta)] \times 10^{-27}$ cm^2/sterad with coefficients derived by the method of least squares.

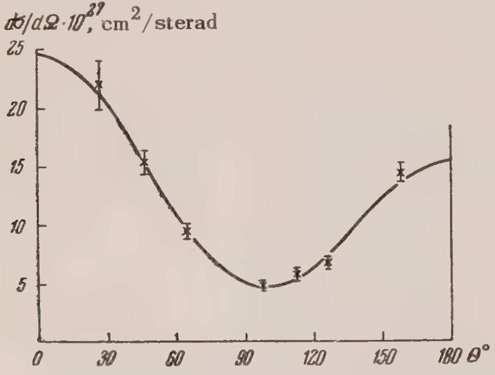

FIG. 14. Angular distribution of 240 mev π^+-mesons scattered from hydrogen. The full curve is for a distribution of the form $d\sigma/d\Omega = [10.1 + 4.6\,P_1(\cos\theta) + 9.9\,P_2(\cos\theta)] \times 10^{-27}$ cm^2/sterad with coefficients derived by the method of least squares.

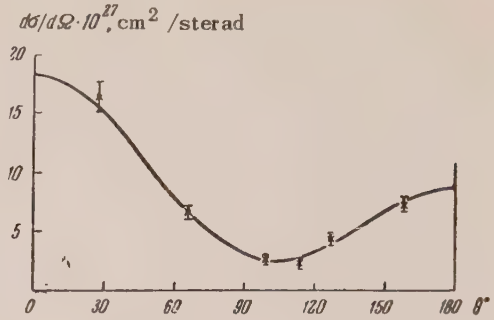

FIG. 15. Angular distribution of 270 mev π^+-mesons scattered from hydrogen. The full curve is for a distribution of the form $d\sigma/d\Omega = [6.5 + 4.7\,P_1(\cos\theta) + 7.1\,P_2(\cos\theta)] \times 10^{-27}$ cm^2/sterad with coefficients derived by the method of least squares.

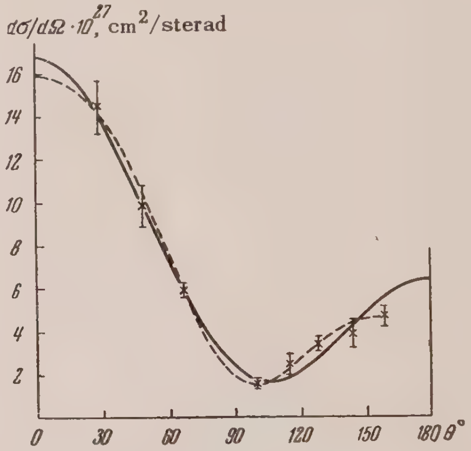

FIG. 16. Angular distribution of 307 mev π^+-mesons scattered from hydrogen. The full and the dashed curves correspond, respectively, to distributions of the form $d\sigma/d\Omega = 5.5 + 5.1\,P_1(\cos\theta) + 6.2\,P_2(\cos\theta) \times 10^{-27}$ cm^2/sterad and $d\sigma/d\Omega = 5.5 + 5.2\,P_1(\cos\theta) + 6.1\,P_2(\cos\theta) + 0.4\,P_3(\cos\theta) - 1.3\,P_4(\cos\theta)$ with coefficients derived by the method of least squares.

squares, of the approximate function with five free parameters.

The dashed curve on Fig. 16 represents the distribution

$$d\sigma/d\Omega = [5.49 + 5.22P_1(\cos\theta) + 6.05P_2(\cos\theta) + 0.43P_3(\cos\theta) - 1.34P_4(\cos\theta)]\cdot 10^{-27}\ \text{cm}^2/\text{sterad}.$$

B. TOTAL CROSS SECTIONS

The magnitudes of the total cross sections are given in Table 10. The errors shown are standard

TABLE 10. Total interaction cross section of π^+–mesons with hydrogen.

Meson energy, in mev	$\sigma_t\,(\pi^+,\,p)$, 10^{-27} см^2	$\int (d\sigma/d\Omega)\,d\Omega=4\pi a_0$, 10^{-27} см^2	$\int (d\sigma/d\Omega)\,d\Omega = 4\pi a_0$, 10^{-27} см^2
176±4	193±6*	199.4±4.9	
200±5		177.9±3.7	
240±6	125.6±2,5	127.2±3.6	
270±6	85.2±3,0	81.3±3.1	
307±6	65.7±2,2	68.5±2.4	69.0±2.6

* Measurement made earlier at an energy of 174 mev (see Ref. 11).

deviations and include the uncertainty in the corrections, taking into account the finite solid angle subtended by the last counter, errors in counting in the registering equipment and the admixture of μ-mesons. The contribution of the integral

$$\int_0^{\theta_1} \{(d\sigma_\pi/d\Omega)_{\text{lab}} + (d\sigma_p/d\Omega)_{\text{lab}}\}\,d\Omega$$

to the total cross section constituted from 4×10^{-27} to 6×10^{-27} cm^2 with an uncertainty of the order of 1×10^{-27} cm^2 for all energies. The third and fourth column in the same table show the value of $\int (d\sigma/d\Omega)\,d\Omega$ determined by the integration of the functions

$$a_0 + a_1P_1(\cos\theta) + a_2P_2(\cos\theta) \quad \text{and}$$

$$a'_0 + a'_1P_1(\cos\theta) + a'_2P_2(\cos\theta) + a'_3P_3(\cos\theta) + a'_4P_4(\cos\theta)$$

with coefficients which were given in Tables 8 and 9.

Together with the energy dependence of $\sigma_t(\pi^+,p)$ found by the method of beam attenuation in this and in previous work,[11] Fig. 17 presents data found by integrating differential cross sections. As can be seen, the agreement of the magnitudes of the total cross sections appears good independently of the method used. There are some discrepancies only in the neighborhood of 200 mev. The comparison of these results with the data from other experiments[14] is carried out in the next paper which is devoted to discussion and interpretation of the results.[15]

The authors are pleased to thank B. S. Neganov who helped in obtaining the meson beam, and also V. V. Krotov for help with the work.

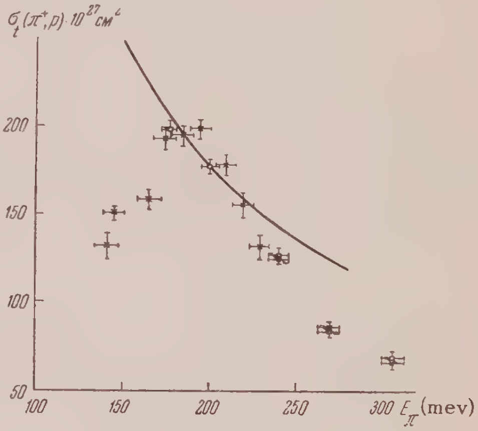

FIG. 17. Total interaction cross section of π^+-mesons with hydrogen from: x –the method of attenuation of the meson beam; ☐ –the integration of the angular distributions. The full curve shows the dependence of $8\pi\lambda\!\!\!^{-2}$ on energy.

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Investigation of Stability of Electron Motion in Cyclic Accelerators when Quantum Effects are Included*

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The general case of motion of electrons in cyclic accelerators has been investigated, taking account ot quantum effects, when axial as well as radial oscillations are possible. We show that the quantum effects can be included by quantizing the adiabtaic invariants according to the Bohr-Sommerfeld method. The effect of quantum fluctuations on radial-phase oscillations in a synchroton is also investigated. Finally, the problem of quantum excitation of macroscopic oscillations is discussed.

1. ADIABATIC INVARIANTS AND EQUILIBRIUM ORBIT

IN the present paper, we want to investigate electron motion in a magnetic field according to quantum theory, and include the possibility that axial as well as radial oscillations can be produced. In this case it is simplest to solve the problem in cylindrical coordinates $r=\sqrt{x^2+y^2}$, z, φ.

We shall first consider the motion of electrons in cylindrical systems like the betatron, where the variation of the magnetic field H over the region around the stationary orbit ($r=R_0$ = const, $z=0$) is given by

$$H = \mathrm{const}\cdot r^{-q}, \tag{1}$$

and its average value satisfies the Wideroe condition

$$\overline{H}(R_0) = \frac{2}{R_0^2}\int\limits_0^{R_0} rH(r)\,dr = 2H(R_0), \tag{2}$$

which has been investigated in detail by Terletskii.[5] In addition, the magnetic field H must satisfy the equations:

$$\mathrm{div}\,\mathbf{H} = 0, \qquad \mathrm{curl}\,\mathbf{H} = 0. \tag{3}$$

In order to satisfy conditions (2) and (3), the vector potential is taken in the form

$$A_x = -{}^1/_2\, y\overline{H}, \quad A_y = {}^1/_2\, x\overline{H}, \tag{4}$$

and the average value $\overline{H}$ set equal to

$${}^1/_2\,\overline{H}(r,z) \tag{5}$$

$$= \frac{H(r)}{(2-q)}\sum_{i=0}^{\infty} b_i\left(\frac{z^2}{r^2}\right)^i + \frac{H(r)(1-q)}{(2-q)}\left(\frac{r}{R_0}\right)^{q-2},$$

$$b_i = (-1)^i(q-2)^2q^2(q+2)^2 \tag{6}$$

$$\ldots(q+2i-2)^2/(2i)!\,(q-2)(q+2i-2).$$

$H(r)$ is given by formula (1). The last term on the right of (5) is chosen so that, on the one hand, it does not affect the value of the magnetic field in the region of the stable orbit ($z=0$), since

$$H(r) = \frac{1}{r}\frac{\partial\, {}^1/_2\, r^2\overline{H}}{\partial r}, \tag{7}$$

and on the other hand so that conditions (1) and (2) are satisfied for $r=R_0$ and $z=0$. It will be sufficient to take the first two terms in the expansion of the sum in (5):

$${}^1/_2\,\overline{H}(r,z) \approx \frac{H(r)}{(2-q)}\left[1+\frac{q(2-q)}{2}\frac{z^2}{r^2}\right] \tag{8}$$

$$+ \frac{H(r)(1-q)}{(2-q)}\left(\frac{r}{R_0}\right)^{q-2}.$$

In our case, the Lagrangian and the generalized momenta are

$$L = -mc^2[1-c^{-2}(\dot{r}^2+r^2\dot{\varphi}^2+\dot{z}^2)]^{1/2} \tag{9}$$

$$-(e/2c)\overline{H}r^2\dot{\varphi},$$

$$p_\varphi = (e/c)r^2(H-{}^1/_2\,\overline{H}), \tag{10}$$

$$p_r = (E/c^2)\dot{r}, \qquad p_z = (E/c^2)\dot{z}$$

Here we have made use of the fact that for the stationary orbit

$$\beta E = eHR_0, \tag{11}$$

and the electron energy is

$$E = mc^2/\sqrt{1-\beta^2}, \tag{12}$$

where $c\beta$ is the electron velocity and m its rest mass.

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* The present paper is a continuation of a series of papers[1-4] by the authors on the quantum theory of the radiating electron, which we shall cite as I-IV.

We shall show that the electron trajectory, including quantum effects, can be found using the Bohr quantum theory. The final results agreee in first approximation with the results of rigorous quantum theory. This method is very convenient for practical computations when the energy is not too high, since it is very simple and descriptive.

We introduce the three adiabatic invariants associated with the azimuthal quantum number $l = 0, +1, +2 \ldots$, the radial quantum number $s = 0, 1, 2 \ldots$, and the axial quantum number $k = 0, 1, 2 \ldots$

$$\oint p_\varphi d\varphi = 2\pi\hbar l, \qquad \oint p_r dr = 2\pi\hbar s, \qquad \oint p_z d_z = 2\pi\hbar k. \tag{13}$$

According to the Ehrenfest adiabatic principle, the adiabatic invariants remain unchanged during slow variations of the magnetic field. In particular, we see from (10) and (13) that on the stationary orbit $[\,H(R_0) = \tfrac{1}{2}\bar{H}(R_0)\,]$ in the absence of radial and axial oscillations all the adiabatic invariants become zero, i.e., $l = s = k = 0$ (the "ideal orbit").

2. OSCILLATION OF THE ELECTRON ABOUT THE INSTANTANEOUS EQUILIBRIUM ORBIT, ACCORDING TO CLASSICAL THEORY

We shall investigate the stability of the motion when the adiabatic invariants are different from zero.

In the relativistic case, the energy is related to the momenta by

$$E^2/c^2 - m^2c^2 = p_r^2 + p_z^2 + V(r, z), \tag{14}$$

where the function $V(r, z) = (\hbar l/r + er\bar{H}/2c)^2$ can be considered to be the expression for some effective potential energy (cf. Ref. 6). The radius R of the instantaneous equilibrium orbit, for which there are no radial oscillations ($s=0$) and no axial oscillations ($k=0$) can be found from the equations

$$\left.\frac{\partial V(r, z)}{\partial r}\right|_{r=R} = \frac{\partial V(r, z)}{\partial z} = 0. \tag{15}$$

We thus obtain

$$z = 0, \quad R = \left[\frac{nc\hbar\,(2-q)}{eH(R_0)\,R_0^q\,(1-q)}\right]^{1/(2-q)} = \left[R_0^{2-q} + \frac{lc\hbar\,(2-q)}{eH(R_0)\,R_0^{l}\,(1-q)}\right]^{1/(2-q)} \approx R_0 + \frac{lc\hbar}{eH(R_0)\,R_0\,(1-q)}, \tag{16}$$

where the quantum number

$$n = l + l_0, \quad l_0 = R_0^2 eH(R_0)(1-q)/c\hbar(2-q)$$

is not an adiabatic invariant since it follows from (16) that, even when $\hbar l \neq 0$, with increasing magnetic field the radius R of the instantaneous equilibrium orbit approaches the radius R_0 of the stationary orbit ($R \to R_0$).

When $s \neq 0$ and $k \neq 0$, using Eqs. (10) and (14) we get the following approximate equations describing the radial and axial oscillations:

$$\frac{E}{c^2}\frac{d^2}{dt^2}(r-R) + \frac{E}{c^2}\omega_1^2(r-R) = 0, \tag{17}$$

$$\frac{E}{c^2}\frac{d^2}{dt^2}z + \frac{E}{c^2}\omega_2^2 z = 0, \tag{18}$$

where, for sufficiently large values of H, we may set [7]

$$\omega_1 = [(c^4\hbar^2/E^2)(e^2H^2/c^2\hbar^2)(1-q)]^{1/2} = \omega_0\sqrt{1-q}, \quad \omega_2 = \omega_0\sqrt{q}, \tag{19}$$

where $\omega_0 = c/R$ is the angular velocity of rotation of the electron in a circular orbit. From this is clear that the oscillations will be stable for $0 < q < 1$. We use (13) to determine the amplitudes A and B. We then get

$$A^2 = 2\hbar cs/e\sqrt{1-q}H = A_0^2 H(0)/H, \tag{20}$$

$$B^2 = 2\hbar ck/e\sqrt{q}H = B_0^2 H(0)/H, \tag{21}$$

where $H(0)$ is the magnetic field at $t=0$. It is then clear that as the magnetic field is increased adiabatically, the amplitudes A and B of the radial and axial oscillations gradually decrease. Substituting the solutions of equations (17) and (18) in (14), we find the expression for the energy of the electron

$$E = \left\{e^2H^2(R)\,R^{2q}\left[\frac{nc\hbar\,(2-q)}{eH(R)\,R^q\,(1-q)}\right]^{2(1-q)/(2-q)} + eH(R)\sqrt{1-q}\,2c\hbar s + eH(R)\sqrt{q}\,2c\hbar k + m^2c^4\right\}^{1/2}.$$

Differentiating with respect to the adiabatic invariants, we again get the values of the circular frequencies

$$\omega_0 = \partial E / \partial \hbar l = c / R,$$

$$\omega_1 = \partial E / \partial \hbar s = \sqrt{1 - q}\, \omega_0,$$

$$\omega_2 = \partial E / \partial \hbar k = \sqrt{q}\, \omega_0.$$

3. CRITERION FOR APPEARANCE OF QUANTUM EFFECTS

As was shown by Ivanenko and one of the authors [8] (cf. also Reference 9), the spectral angular distribution of the radiation is given, in the classical case, by the expression

$$W_\nu (\theta)\, d\nu \sin \theta d\theta = \frac{e^2\nu^2 c}{3\pi^2 R^2}\left\{\varepsilon^2 K_{2/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right) + \varepsilon \cos^2 \theta K_{1/3}\left(\frac{\nu}{3}\varepsilon^{3/2}\right)\right\} d\nu \sin \theta d\theta. \tag{22}$$

Integrating this expression over angles, we get a formula which quite accurately describes the spectral distribution of intensity of the radiation over the whole frequency range:

$$W_\nu d\nu = \frac{9\sqrt{3}}{8\pi} W_{\mathrm{cl}}\, y dy \int_y^\infty dx K_{5/3}(x), \tag{23}$$

where $d\nu$=1, ϵ=1 $-\beta^2 \sin^2 \theta$ and the quantity y is proportional to the harmonic number of the radiation: y= 2/3 ν (mc ^{2} / E)^3. Integrating (23) over all frequencies, we get

$$\int_0^\infty W_\nu d\nu = (2e^2 c / 3R^2)\,(E / mc^2)^4 = W_{\mathrm{cl}}.$$

The integral intensity of the radiation, taking quantum corrections into account is

$$W = W_{\mathrm{cl}}\, \{1 - (55\sqrt{3} / 16)\,(\hbar / mcR_0)(E / mc^2)^2\}.$$

(cf. I and II; later this formula was also obtained in Reference 10). From this we see that quantum corrections to the integral intensity will be comparable to the corresponding classical quantities only in the region of very high energies, $E \sim E_{1/2}$, where

$$E_\mu = mc^2 (mcR_0 / \hbar)^\mu. \tag{24}$$

This condition (μ = ½) for the appearance of quantum corrections refers to the quantum number l or n. Despite the limitations on this condition, some authors assume that for $E << E_{1/2}$ we can in general neglect quantum corrections (cf., for example, Ref. 11), and assert that all attempts to set a more stringent condition for quantum corrections than the condition $E \sim E_{1/2}$ will fail. We cannot agree with this viewpoint. For this purpose, let us study the change of the other adiabatic invariants $\hbar s$ and $\hbar k$ as a result of radiation.

The amplitudes of harmonic oscillations can change non-adiabatically if: a) there is a sudden change in the position of the center of oscillation, or b) some momentum is suddenly imparted to the oscillating point. Suppose a material point is carrying out a harmonic oscillation along the x-axis. Then the square of the oscillation amplitude can be found from the relation $D_0^2 = x^2 + p^2 c^4 / E^2 \omega^2$, where p= (E / c^2) $\dot{x}$ is the momentum of the particle. The changes in the quantities x and p are given by

$$x = D_0 \sin (\omega t + \varphi),\ p = (E\omega / c^2)\, D_0 \cos (\omega t + \varphi),$$

where φ is a phase which, in general, depends on the initial conditions.

Let us suppose that at a certain instant of time the center of oscillation is shifted by an amount Δx, or the momentum is changed by an amount Δp. Then the square amplitude of the oscillation will be

$$\begin{aligned} D^2 &= (x - \Delta x)^2 + (p + \Delta p)^2 c^4 / E^2\omega^2 \\ &= D_0^2 - 2D_0 \sin (\omega t + \varphi)\, \Delta x \\ &\quad + \Delta x^2 + (c^4 / E^2\omega^2)\, [\Delta p^2 \\ &\quad + 2\Delta p\, D_0\, (E\omega / c^2) \cos (\omega t + \varphi)]. \end{aligned}$$

If we assume that the excitation of oscillations occurs statistically independently (cf. Ref. 12)*, then, by averaging the last expression over φ, we get

$$(\Delta D)^2 = (\Delta x)^2 + (c^4 / E^2\omega^2)\, (\Delta p)^2. \tag{25}$$

From formula (16) we see that the radiation of a photon with energy $\hbar\nu\omega_0$ leads to a decrease of the radius of the trajectory by the amount

$$\Delta R = \frac{1}{2 - q}\,\frac{\nu}{n}\, R = \frac{1}{1 - q}\,\frac{\Delta E}{E}\, R, \tag{26}$$

where $\Delta E \approx \nu c\hbar / R$. We can get an analogous expression for the change in radius by introducing, on the right side of Eq. (17) which describes the radial oscillations in the classical case, the fluctuation force

$$F_r = \frac{1}{R}\left[\sum_i \mu(t,\ t_i)\, \hbar\omega_i - \int_0^t W dt\right]$$

---

* This result will be established more rigorously later, using quantum mechanics (cf. Section 4).

(t_i is the time of emission, $\mu = 1$ for $t > t_i$, $\mu = 0$ for $t < t_i$). On the other hand, as we see from (20), quantum radial transitions can occur if the square amplitude of the oscillation changes by at least the amount $\Delta(A^2) \sim \hbar c/eH = R^2/n$ ($\Delta s = 1$). Thus we obtain the condition $(\Delta R)^2 > \Delta(A^2)$ or $\nu^2/n \gg 1$ for production of radial oscillations as a result of quantum fluctuations of the radius. Substituting the harmonic number ν corresponding to the maximum of the radiation, $\nu \sim (E/mc^2)^3$, and setting n equal to $n \sim RE/\hbar c$, we find the condition for appearance of radial quantum oscillations to be $E > E_{1/5}$, where $E_{1/5}$ is given by formula (24) with $\mu = 1/5$. This condition was already found in the first papers on the quantum theory of the radiating electron[13] (cf. also I and Ref. 14).

It is just as easy to get the condition for appearance of axial quantum oscillations. Axial oscillations are produced as a result of the recoil of the electron when it emits a photon. When a photon is emitted, the z component of the momentum changes by an amount $\Delta p_z = (\hbar\omega_0/c)\,\nu\cos\theta$. According to (25), the increase in the amplitude of axial oscillation due to emission of one quantum is

$$(\Delta B_1)^2 = (R^2/q)(\Delta E/E)^2\cos^2\theta \sim R^2\nu^{4/3}/n^2. \quad (27)$$

We note that (27) can also be obtained from the classical equation (18) for the axial oscillations, by substituting on the right the corresponding fluctuation force

$$F_z = \sum_i \Delta p_z\,\delta(t - t_i).$$

On the other hand, for a minimal change of the adiabatic invariant ($\Delta k = 1$), the increase of the square amplitude is equal, according to (21) to

$$\Delta(B^2) \sim \hbar c/eH = R^2/n. \quad (28)$$

From (27) and (28) we see that quantum corrections for the axial oscillations must be considered when $(\Delta B_1)^2 > \Delta(B^2)$, i.e. $\nu^{4/3}/n > 1$ or $E > E_{1/3}$, where $E_{1/3}$ is given by (24) with $\mu = 1/3$. This condition was given in the first edition of the monograph of Ivanenko and one of the authors.[15]

4. DEVELOPMENT OF RADIAL AND AXIAL OSCILLATIONS ACCORDING TO QUANTUM THEORY

According to the criteria which we have established in the preceding paragraph, quantum effects should first influence the radial oscillations (when $E \sim E_{1/5}$), then the axial oscillations (when $E \sim E_{1/3}$), and will only affect the total intensity of the radiation in the region of very high energies (when $E \sim E_{1/2}$).

Quantum excitation of radial oscillations should begin to play an important role in the energy region $E > E_{1/5}$, since according to the classical theory the amplitude of oscillation then tends to zero.

As we see from formulas (20), (21), (26) and (27), when a photon is radiated the quantum numbers n, s, and k change by the amounts

$$\Delta l = l' - l = -\nu, \quad (29)$$

$$\Delta s = \frac{1}{2(1-q)^{3/2}}\frac{\nu^2 c\hbar}{R^2 eH(R)},$$

$$\Delta k = \frac{1}{2\sqrt{q}}\frac{\nu^2 c\hbar}{R^2 eH(R)}\cos^2\theta,$$

But according to classical theory, the quantum numbers s and k should remain constant. The expressions (29) were obtained by a semiclassical method (since in considering the radiation of an individual photon, we averaged over the phase φ). We shall show that a more rigorous quantum treatment will lead us to the same expression (29). Actually we can treat the oscillations along r and z as oscillations at non-relativistic velocities, but with the relativistic mass E/c^2, and having frequencies $\omega_1 = \sqrt{1-q}\,c/R$ and $\omega_2 = \sqrt{q}\,c/R$ respectively. Thus the wave functions will have the forms

$$\Psi_s = \left(\frac{\lambda_1}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^s s!}}e^{-\lambda_1(r-R)^2/2}H_s[\sqrt{\lambda_1}(r-R)],$$

$$\Psi_k = \left(\frac{\lambda_2}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^k k!}}e^{-\lambda_2 z^2/2}H_k(\sqrt{\lambda_2}\,z),$$

$$\lambda_1 = \omega_1 E/\hbar c^2 = \sqrt{1-q}\,eH(R)/c\hbar,$$

$$\lambda_2 = \omega_2 E/\hbar c^2 = \sqrt{q}\,eH(R)/c\hbar,$$

while the radius of the instantaneous equilibrium orbit $R = R(l)$ is given by equation (16). Let us find the probability of transition $W_{ss'}$ of an electron from a state s to state s', with a change of azimuthal quantum number from n to $n' = n - \nu$. The transition probability is equal to the square of the matrix element

$$W_{ss'} = \left| \left(\frac{\lambda_1}{\pi 2^{s+s'} s!\, s'!} \right)^{1/2} \int_0^\infty dr \exp\left\{ -\frac{\lambda_1}{2} [r - R(n)]^2 \right.\right.$$

$$\left.\left. - \frac{\lambda_1}{2} [r - R(n-\nu)]^2 \right\} H_s [\sqrt{\lambda_1} (r - R(n))]\ H_{s'} [\sqrt{\lambda_1} (r - R(n-\nu))] \right|^2$$

$$= \left| \frac{e^{-\xi_1/2}}{\sqrt{\pi 2^{s+s'} s!\, s'!}} \int_{-\infty}^{\infty} dx e^{-x^2} H_s \left(x - \sqrt{\frac{\xi_1}{2}} \right) H_{s'} \left(x + \sqrt{\frac{\xi_1}{2}} \right) \right|^2 = I_{ss'}^2 (\xi_1),$$

$$\xi_1 = {}^1/_2 \lambda_1 [R(n) - R(n-\nu)]^2 = \nu^2 / 2n (2-q) \sqrt{1-q}.$$

The functions $I_{ss'}(\xi_1)$ are related to the Laguerre polynomials $Q_s^{s'-s}(\xi_1)$:

$$I_{s's}(\xi) = \frac{\xi^{(s'-s)/2} e^{-\xi/2}}{\sqrt{s!\, s'!}} Q_s^{s'-s}(\xi). \tag{30}$$

Thus, in a single transition, the change of the quantum number s is

$$\Delta s = \sum_{s'=0}^{\infty} (s'-s)\ I_{ss'}^2 (\xi_1) \tag{31}$$

$$= \xi_1 = c\nu^2\hbar / 2 (1-q)^{3/2} eHR^2.$$

Earlier we obtained this last expression by a semiclassical method [cf. Eq. (29)].

By a similar method we get the transition probability $W_{kk'}$ from a state k to state k', when the z component of the photon momentum is

$$\Delta p_z = \hbar (\omega_0 / c) \nu \cos\theta;$$

$$W_{kk'} = \left| \left(\frac{\lambda_2}{\pi 2^{k+k'} k!\, k'!} \right)^{1/2} \int_{-\infty}^{\infty} dz \right.$$

$$\times \exp\left\{ -\lambda_2 z^2 - iz \frac{\omega_0}{c} \nu \cos\theta \right\}$$

$$\left. \times H_k (\sqrt{\lambda_2} z) H_{k'} (\sqrt{\lambda_2} z) \right|^2$$

or

$$W_{kk'} = I_{kk'}^2 (\xi_2),$$

where $\xi_2 = \omega_0^2 \nu^2 \cos^2\theta / 2c^2 \lambda_2$, and the functions $I_{kk'}(\xi_2)$ are given by (30). When a photon is emitted, as a result of the recoil the adiabatic invariant k which measures the amplitude of the axial oscillations changes by an amount

$$\Delta k = \sum_{k'=0}^{\infty} (k'-k)\ I_{kk'}^2 (\xi_2) \tag{32}$$

$$= \xi_2 = \nu^2 c\hbar \cos^2\theta / 2 \sqrt{q} eHR^2,$$

which coincides precisely with the corresponding expression (29) which we found by a semiclassical method. Thus the average over phases in the semiclassical method gives the quantum result, which justifies the general theorem of statistical independence of emission of photons in the quantum case.

As we see from formulas (31) and (32), the change of the adiabatic invariants s and k per unit time as a result of quantum transitions will be

$$\frac{dl}{dt} = - \int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{\nu W_\nu(\theta)}{\hbar \nu \omega_0},$$

$$\frac{ds}{dt} = \int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{c\nu^2\hbar}{2(1-q)^{3/2} eHR^2} \frac{W_\nu(\theta)}{\hbar\nu\omega_0}, \tag{33}$$

$$\frac{dk}{dt} = \int_0^\infty d\nu \int_0^\pi \sin\theta d\theta \frac{c\nu^2\hbar \cos^2\theta}{2\sqrt{q}\, eHR^2} \frac{W_\nu(\theta)}{\hbar\nu\omega_0},$$

where the quantity $W_\nu(\theta)$ is given by (22). Introducing the variable

$$x = (\nu/3) \varepsilon^{3/2} = (\nu/3)(1 - \beta^2 \sin^2\theta)^{3/2}$$

and carrying out the integrations (cf. II), we obtain expressions giving the total change of the radius and also of the amplitudes of radial and free axial oscillations:

$$R^{2-q} = R_0^{2-q} - \frac{2}{3} \frac{e^2 c}{eH} \frac{2-q}{1-q} \int_0^t \left(\frac{E}{mc^2} \right)^4 \frac{1}{R} dt, \tag{34}$$

$$A^2 = A_0^2 \frac{H(0)}{H(t)} \tag{35}$$

$$+ \frac{55}{24\sqrt{3}} \frac{e^2\hbar R}{(1-q)^2 mE} \int_0^t \left(\frac{E}{mc^2} \right)^6 \frac{1}{R^2} dt,$$

$$B^2 = B_0^2 \frac{H(0)}{H(t)} + \frac{13}{24\sqrt{3}} \frac{e^2\hbar R}{qmE} \int_0^t \left(\frac{E}{mc^2} \right)^4 \frac{1}{R^2} dt. \tag{36}$$

The first term in Eqs. (35) and (36) corresponds to the classical contraction and the second to the quantum broadening. Formula (35) for the radial oscillations was obtained by us earlier, both for a uniform field (cf. III, $q = 0$)*, and for a field having axial symmetry (cf. IV, $q \neq 0$). In the present paper we have generalized this formula to the case where axial oscillations are also possible, and have also determined the influence of quantum effects on the magnitude of the axial oscillations. From the last formulas we see that quantum effects should have a very strong influence only on the free radial oscillations.

As we see from formulas (20) and (35), because of the quantum character of the radiation the percentage increase of the mean square radial deviation from the instantaneous equilibrium orbit $\overline{\Delta R^2_{qu}} = \overline{x^2} = 1/2A^2$ will vary according to the law (cf. also III and IV)

$$\left(\frac{\Delta R_{\text{qu.}}}{R}\right)^2 = \frac{55\sqrt{3}}{48}\,\frac{e^2\hbar c}{(1-q)^2\, mER}\int_0^t \left(\frac{E}{mc^2}\right)^6 \frac{1}{k^2}\,dt. \tag{37}$$

5. RADIAL-PHASE OSCILLATIONS IN A SYNCHROTRON WHEN QUANTUM EFFECTS ARE INCLUDED

In the preceding paragraphs we have shown that fluctuations of the radius, which are associated with the quantum character of the emission, lead to excitation of a special type of radial oscillations. Therefore, for the motion of an electron in a synchrotron, we must still estimate the effect of these fluctuations on the so-called radial-phase oscillations.

In a synchrotron, the increase in energy of the electron results from passage of the electron through an accelerating gap. In a single turn the energy increases by the amount $eV\sin\varphi$ where V is the amplitude of the voltage across the accelerating gap, and φ is the phase at which the electron crosses the gap. On the other hand, in one turn the electron loses the energy $I = (4\pi e^2/3R)\,(E/mc^2)^4$ as a result of radiation. Thus the average increase per unit time is

$$dE/dt = \omega\,(eV\sin\varphi - I)/2\pi, \tag{38}$$

where ω is the angular velocity of rotation of the electron. We shall denote quantities referring to the instantaneous equilibrium orbit by the index s. These quantities are related by the equations

$$E_s = eH_sR_s, \quad (\omega_s/2\pi)\,\Delta E_s = (ec/\omega_s)\,\dot{H}_s.$$

The increase in energy of the electron will be given by

$$\begin{aligned}(\omega_s/2\pi)\,\Delta E_s &= dE_s/dt \\ &= \omega_s\,(eV\sin\varphi_s - I_s)/2\pi.\end{aligned} \tag{39}$$

We note that in the extreme relativistic region the energy loss per turn reaches values far exceeding the average energy given to the electron per turn, i.e.,

$$\Delta E_s \ll eV\sin\varphi_s \approx I_s. \tag{40}$$

Denoting the deviation from the equilibrium phase by $\psi = \varphi - \varphi_s$, we have

$$\begin{aligned}\omega - \omega_s &= \dot{\psi}, \\ \Delta R/R_s &= \Delta E/(1-q)\,E_s = -\dot{\psi}/\omega_s,\end{aligned} \tag{41}$$

where $\Delta R = R - R_s$, $\Delta E = E - E_s$. Subtracting (39) from (38), we get

$$\frac{d}{dt}\Delta E = \frac{eV\cos\varphi_s}{2\pi}\,\omega_s\psi - \frac{I - I_s}{2\pi}\,\omega_s + \frac{\Delta E_s}{2\pi}\,\dot{\psi}. \tag{42}$$

Using the fact that in the ultrarelativistic case, at a given time, $E = eHR = \text{const}\cdot R^{1-q}$, i.e., $I = \text{const}\cdot R^{3-4q}$, we find

$$I - I_s = -(3-4q)\,I_s\dot{\psi}/\omega_s.$$

In addition, using (40) we can drop the last term on the right of (42) and set $eV\cos\varphi_s \approx I_s\cot\varphi_s$. Then we get the equation for the radial-phase oscillations

$$\ddot{\psi} + \gamma\dot{\psi} + \Omega^2\psi = 0. \tag{43}$$

The damping coefficient γ and the angular frequency of the radial-phase oscillations, Ω, are equal respectively to

$$\gamma = (W_s/E_s)(3-4q)/(1-q),$$

$$\Omega^2 = \omega_s W_s \operatorname{ctg}\varphi_s/(1-q)\,E_s,$$

where the quantity $W_s = I_s(c/2\pi R_s)$ is the energy loss through radiation, per unit time, for an electron on the stationary orbit.

From this it is clear that the radial-phase oscillations will be damped for $q < 3/4$; if $q > 3/4$ the motion will be unstable. From equations (41) and (43) we find the following differential equation for ΔR:

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* For a uniform field, Eq. (35) was obtained rigorously in III, using the Dirac equation.

$$d^2\Delta R / dt^2 + \gamma d\Delta R / dt + \Omega^2 \Delta R = 0. \tag{44}$$

The solution of this equation has the form:

$$\Delta R = e^{-\gamma t/2} \cos \Omega t \Delta A. \tag{45}$$

For high energies, the quantity ΔA will be the oscillation amplitude which results from the quantum character of the radiation. We then get for the mean square percentage shift in radius the value:

$$(\Delta R / R_s)^2 = {}^1/_2 (\Delta A / R_s)^2 e^{-\gamma t}. \tag{46}$$

Suppose that the radiation occurs at some time t_i. Then we get for the square of the ratio $\Delta R/R_s$ the value

$$\left(\frac{\Delta R}{R_s}\right)^2 = \frac{1}{(1-q)^2} \frac{(\Delta E_\gamma)^2}{E_s^2} = \frac{1}{2} \sum_\nu e^{-\gamma (t-t_i)} W_\nu \, dt, \tag{47}$$

where $\Delta E_\gamma = \hbar \nu c/R_s$, while W_ν is given by (23). Summing the right side of (47) over ν, and assuming that during the damping time $\tau = 1/\gamma$ the energy E_s can be taken to be constant, we get the equation found by Sands[16],

$$\left(\frac{\Delta R}{R}\right)^2 = \frac{55}{48\sqrt{3}} \frac{1}{(1-q)^2} \frac{\hbar}{mcR_s^3} \frac{e^2}{mc} \left(\frac{E_s}{mc^2}\right)^5 \tau. \tag{48}$$

In this case, Eq. (37) should be used for $t < \tau$, and its limiting form [cf. Eq. (48)] for $t \gg \tau$.

We note that relation (48) can be obtained from our more general formula (37) [cf. also III and IV] if we limit the duration of the free betatron oscillations to the damping time τ.

In conclusion, we note that the amplitudes of the radial oscillation and radial momentum will be of order

$$A \sim \sqrt{c\hbar s/eH}, \tag{49}$$

$$P \sim (E\omega/c^2) A \approx \sqrt{eH\hbar s/c}.$$

If the quantum number s changes by unity as the result of emission of radiation, we get an uncertainty in these amplitudes equal to

$$\Delta A \approx (\partial A/\partial s) \Delta s \sim A/s, \quad \Delta P \sim P/s, \tag{50}$$

since it is impossible by present theory to determine when the emission occurred. In the classical approximation $E \ll E_{1/5}$, the value of s does not change, so that this indeterminacy cannot occur. But in the quantum case $E \gg E_{1/5}$, a change of the quantum number s will occur several times in the course of a single period of oscillation. The change in s leads to indeterminacies in the coordinates, Δx and Δp, which cannot be given by (50), since for $s \gg 1$ we would obtain the inequality $\Delta x \Delta p \sim \hbar/s \ll \hbar$, which contradicts the uncertainty relation $\Delta x \Delta p \sim \hbar$. Thus we have two limiting cases giving uncertainties in coordinate and momentum,

$$\Delta x \sim \Delta A \sim A/s, \quad \Delta p \sim \hbar/\Delta x \sim P$$

$$\Delta p \sim \Delta P \sim P/s, \quad \Delta x \sim \hbar/\Delta p \sim A$$

Neither of the two cases permits a classical approximation. To investigate the quantum excitation of macroscopic radial oscillations, forming a sort of "macro-atom", we can use either the rigorous methods of quantum theory as was done in Section 4, or the classical equations (17) and introduce fluctuation forces (cf. Section 3).

In the equivalence of the two methods we are inclined to see a connection between quantum methods and the theory of fluctuations, where there are Markoff processes whose basis is the statistical independence of successive processes (cf., for example, the theory of the Brownian motion, and also Welton's theory which qualitatively explains the Lamb shift by considering fluctuations of the virtual photons). In any case, in the present example both theories actually give the same quantitative results.

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Letters to the Editor

Polarization of Neutrons Scattered by Lead

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IN the scattering of fast neutrons by nuclei of high Z a strong polarization should be observed at small angles due to the interaction of the magnetic moment of the neutron with the electric field of the nucleus. This interaction has the form:

$$H' = -\mu_n (eh/2M^2c^2)\, \sigma[\mathrm{E}\cdot\mathrm{p}] = \xi r^{-3}(\sigma\cdot \mathrm{l});\ \xi = +\mu_n e^2h^2Z/2M^2c^2, \tag{1}$$

where $\mu_n = -1.91$ is the magnetic moment of the neutron, σ is the neutron spin operator, p its momentum, and E the electric field of the nucleus. This effect was first shown by Schwinger[1] who evaluated the resulting polarization. In that work he considered the nucleus either as an absolutely black body or as a hard sphere.

In the present paper the polarization of neutrons was calculated more exactly. The interaction Hamiltonian of the neutron with the nucleus was expressed in the following form:

$$U = \frac{U_0(1+i\zeta)}{1+\exp[(r-r_0)/a]} + H^1 = U_1 + H^1, \tag{2}$$

where $U_0 = -40$ mev, ζ a constant describing the absorption, r_0 the nuclear radius, $2a$ the smear of the nuclear edge, and H^1 is determined by Eq. (1). The first term in the right side of Eq. (2) was written as the sum of two parts:

$$U_1 = U_2 + (U_1 - U_2),$$

where

$$U_2 = \begin{cases} U_0(1+i\zeta) & \text{for } r < r_0, \\ 0 & \text{for } r > r_0. \end{cases} \tag{3}$$

The Schrödinger equation for a neutron in the potentials U_2 was solved exactly, while the additions to the Hamiltonian, $(U_2 - U_1)$ and H^1, were taken into account by perturbation theory. In this way a neutron scattering amplitude was calculated:

$$f(\theta) = f_1(\theta) + (\sigma \mathrm{n})\, f_2(\theta). \tag{4}$$

It should be noted that for scattering angles $\theta \sim 1$–$3°$ the spin amplitude $f_2 = -i2M\,\xi/h^2\theta$ and thus depends only on the nuclear charge [see Eq. (1)].

Knowing the scattering amplitude, we calculated the total cross section for neutrons

$$\sigma_t = (4\pi/k)\,\mathrm{Im}\, f_1(0),$$

the differential cross section in the region $0° < \theta < 30°$; $d\sigma/do = |f_1|^2 + |f_2|^2$ (we are limited to these angles since the optical model describes scattering well only for small angles), and finally the polarization for small θ

$$P = -\mathrm{n}\frac{2\mathrm{Re}\,[f_1(0),\ f_2^*(\theta)]}{|f_1(0)|^2 + |f_2(\theta)|^2} = \mathrm{n}\frac{k\xi M\sigma_t}{\pi h^2 d\sigma(\theta)/do} = P(\theta)\,\mathrm{n}. \tag{5}$$

All the above calculations were carried out for lead, with various choices of the parameters, r_0, ζ, a and neutron energy ϵ:

$r_0 = 7; 7.5$; and 8 (in units of 10^{-13} cm)
$\zeta = 0; 0.01; 0.02; 0.03; 0.05; 0.10$
$\epsilon = 1; 2; 3; 4; 5$ (in mev)
$a = 0; 0.4; 0.55$ (in units of 10^{-13} cm)

Comparing the magnitudes of σ_T and $d\sigma/do$ calculated for various values of the parameters it turns out that for a given neutron energy ϵ the results of the calculation depend most strongly on the nuclear radius r_0, less strongly on the value of the absorption ζ, and finally, least on the smear of the nuclear edge $2a$. The selection of best values for r_0, ζ and a must be made by a comparison of the theoretical results for σ_t and $d\sigma/do$ with the experimental values. It turns out that agreement with the experimental values occurs only for $r_0 = 8$. As illustration, Fig. 1 shows our calculations of σ_t for various parameters. Comparison of the theoretical curves of σ_t (as a function of

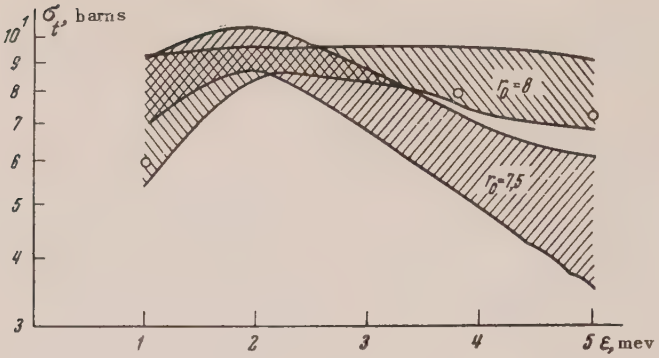

FIG. 1. Graph of total cross section. σ_t lies inside the shaded regions for $r_0 = 8$ and $r_0 = 7.5$ and various values of ζ and a. Experimental values are indicated by the circles.

neutron energy) and the curves of differential cross section $d\sigma/do$ with the experimental values[2] shows that in all cases (with the exception of the case $\epsilon = 1$ mev) the experimental points are between the curves for $r_0 = 7.5 \times 10^{-13}$ cm and $r_0 = 8 \times 10^{-13}$ cm, and usually closer to the latter. Thus the correct value of the radius must lie in the interval $7.5 < r_0 < 8$, and probably closer to $r_0 = 8$ ($r_0 = 8$ corresponds to $R_0 = 1.35$ in the radius formula $R = R_0 \times 10^{-13} A^{1/3}$ cm, and $r_0 = 7.5$ to $R_0 = 1.27$). Since all calculated quantities depend strongly on r_0 and comparitively less on ζ and a, we did not think it reasonable to determine ζ and a from the experimental data. Nevertheless some information about these quantities was obtained. The theoretical value of σ_t for $\epsilon = 1$ mev agrees with the experimental value only for $r_0 = 8$, $\zeta = 0$–0.2 and $a = 0.5$. Values $a=0$ and $\zeta > 0.02$ do not give agreement with experimental values of σ_t for any r_0.

Thus we come to the conclusion that the best agreement with experimental data is achieved for r_0 slightly smaller than 8, and an effective radius edge smear of $2a \sim 1$. These values are in good agreement with those obtained from other sources.[3] With regard to ζ it is only possible to say that for $\epsilon = 1$ mev, $\zeta \sim 0.01$, and with increase in ϵ, apparently does not grow rapidly.

The neutron polarization $P(\theta)$ was calculated from Eq. (5) for $r_0 = 8$, $a = 0.5$ and various values of ζ. It was found that $P(\theta)$ almost does not depend on ζ for neutron energies $\epsilon > 3$ mev, while at 1 mev, where this dependence is stronger, $P(1°)$ changes from 0.5 to 0.8 for variation of from 0 to 0.1. The magnitude of ζ for $\epsilon = 1$ mev is, however, known from the above, and is 0.01 so that we can calculate $P(\theta)$ for neutron energies from 1–5 mev with an accuracy of 10–20%. These curves for the angles $\theta = 1,2,3°$ are shown in Fig. 2.

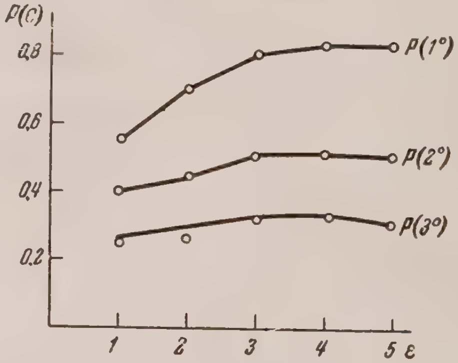

FIG. 2

Comparison of our results to those of Schwinger shows that our calculation leads to a much more rapid decrease of $P(\theta)$ with increasing θ. Schwinger[1] obtained the following values of polarization at an energy $\epsilon = 1$:

$$P(1.5°) = 1;\ P(3°) = 0.8;\ P(6°) = 0.47;\ P(9°) = 0.32.$$

Our calculation gives, at the same energy

$$P(1°) = 0.5;\ P(2°) = 0.45;\ P(3°) = 0.28.$$

It should be noted, as can be seen from Eq. (5), $P(\theta)$ depends only on σ_t and $d\sigma/do$. Since calculations made with our choice of parameters are in good agreement with experimental values of σ_t and $d\sigma/do$, our calculated polarization does not depend on a particular model, and should approach the correct value.

In conclusion, I would like to thank Ia. A. Smorodinskii for his interest in this work.

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Parapositronium Annihilation Probability, with Account of the First Radiative Corrections

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IT is known that parapositronium in the ground state, as a system of even parity, cannot decay into three photons. Therefore it is interesting to calculate the first radiative corrections to the probability for two photon annihilation,[1] since in the next approximation the lifetime of parapositronium is determined just by these radiative corrections and does not depend on the four photon annihilation.

The positronium annihilation probability W for any order of radiative correction is connected with the annihilation probability W_{Free} of the free particles with zero relative velocity by the equation[2]

$$W = [(\bar{\psi}^{E\sigma}(0)\,\psi^{E\sigma}(0))\;/\;(\bar{\psi}^{\mathcal{E}\sigma}_{\text{Free}}(0)\,\psi^{\mathcal{E}\sigma}_{\text{Free}}(0))]\,W_{\text{Free}} \quad (1)$$

where $\psi^{E\sigma}(x)$ is the wave function (in relative coordinates) of the positronium in the ground state, and which satisfies a Bethe-Salpeter equation with the possible annihilation of the particles (Ref. 2; see also Ref. 3). $\psi^{\mathcal{E}\sigma}_{\text{Free}}(x)$ is the wave function of the free particles which turns out to have the same implicit set of functions as $\psi^{E\sigma}(x)$ (but with energy $\mathcal{E} = E + \epsilon$, where $\epsilon > 0$ the binding energy). The sign of σ determines the spin state. In Eq. (1), the quantity $[\bar{\psi}^{E\sigma}(0)\,\psi^{E\sigma}(0)] = \text{Sp}\,[\bar{\psi}^{E\sigma}(0)\,\psi^{E\sigma}(0)]$ should be calculated with the same accuracy as is obtained in the calculation of W_{Free}.

If we limit ourselves to the first radiative corrections, it is not hard to calculate $[\bar{\psi}^{E\sigma}(0)\psi^{E\sigma}(0)]$ with the required accuracy to terms including the order e^2 ($\hbar = c = 1$) using the nonrelativistic approximation to the wave function $\psi^{E\sigma}(0)$. In this process we can drop the small terms of the wave functions in $(\bar{\psi}^{E\sigma}(0)\,\psi^{E\sigma}(0)]$ since they are of order v^2_{rel} ($\sim e^4$). It remains to find the corrections of the order e^2 to the large components of $\psi^{E\sigma}(x)$. This can be done with the aid of ordinary perturbation theory[4] if we use the second approximation[5] to the Hamiltonian for the large components of the wave function. Simple calculations show that the first correction of the nonrelativistic value $\psi_{\text{NR}}(0)$ is of the order e^4, and therefore does not need to be taken into account. To the desired degree of approximation we thus have

$$(\bar{\psi}^{E\sigma}(0)\,\psi^{E\sigma}(0)) = |\,\psi_{\text{NR}}(0)\,|^2. \quad (2)$$

The wave function $\psi^{\mathcal{E}\sigma}_{\text{Free}}(x_1, x_2)$ of the free particles used in W_{Free} is an implicit function of the full energy and momentum operators, and the full spin (since the orbital angular momentum is zero) and its projections. Since the spin operator does not include the spatial coordinates of the particle, the spatial and spin variables in $\psi^{\mathcal{E}\sigma}_{\text{Free}}$ separate, and the wave function has the form:*

$$\psi^{\mathcal{E}\sigma}_{\text{Free}}(x_1 x_2) = \Phi^{\sigma}(K)\,e^{-iK(x_1+x_2)/2}, \quad (3)$$

where the main component of the total particle momentum K is $K_0 = \mathcal{E} = 2m$, and $K \to 0$ in the center-of-mass system. Therefore, $[\bar{\psi}^{\mathcal{E}\sigma}_{\text{Free}}(0)\,\psi^{\mathcal{E}\sigma}_{\text{Free}}(0)]$ in Eq. (1) is a square of the spin function $\Phi^{\sigma}(K)$, which can always be made equal to unity.

The value of W does not change if we sum in Eq. (1) over all four spin states, since $W_{\text{Free}} = 0$ in all spin states with a total spin $S = 1$. In W_{Free} (in the summation over all spin states) let us replace the summation over the complete set of spin functions of the entire spin operator by a summation over the complete set of spin functions which are themselves implicit spin operators of the spin of each particle individually. Then W_{Free} will be the probability, together with the first radiative corrections, for the two photon annihilation of the free particles with momenta $p_1 = p_2 \to m$. Using the results of Ref. 6, which were calculated for the Compton effect, then W_{Free} is written

$$W_{\text{Free}} = W_0 \left[1 - \frac{e^2}{\pi} \frac{\operatorname{Re}(U^{(1)})}{U}\right]. \qquad (4)$$

Here W_0 is the first term of the zeroth approximation, and

$$U(k, \tau) = 4(\tau^{-1} - k^{-1})^2 - 4(\tau^{-1} - k^{-1}) + (k/\tau + \tau/k), \qquad (5)$$

$$U^{(1)} = P(-k, \tau) + P(\tau, -k), \qquad (6)$$

where

$$\begin{aligned} P(a, b) = {} & (1 - 2y \operatorname{cth} 2y) \ln \lambda \cdot U(-a, b) - 2y \operatorname{cth} 2y \,[2h(y) - h(2y)\, U](-a, b) \\ & + [-4y \operatorname{sh} 2y \cdot (ab)^{-1} (2 - \operatorname{ch} 2y) + 2y \operatorname{cth} y]\, h(y) + \\ & + \ln a \left\{4y \operatorname{cth} 2y \left[\frac{4}{ab} \operatorname{ch}^2 y + \frac{a-6}{2b} \frac{1}{\operatorname{ch} 2y} + \frac{4}{a^2} - \frac{1}{a} - \frac{b}{2a} - \frac{a}{b} - 1\right]\right. \\ & \left. + \frac{3b}{2a^2} + \frac{3b}{2a} + \frac{3}{b} + 1 - \frac{7}{ab} + \frac{8}{a} - \frac{8}{a^2} + \frac{2a - b^2 - a^2 b}{2a^2 b(a-1)} - \frac{1}{2b} \frac{2a^2 + b}{(a-1)^2}\right\} \\ & + y^2 \frac{1}{\operatorname{sh}^2 y} \left[\frac{2}{a} - \frac{7}{4} a - \frac{3}{4} \frac{b^2}{a}\right] - 4y \operatorname{th} y \left(\frac{1}{2} - \frac{1}{a}\right) + 4\left(\frac{1}{b} + \frac{1}{a}\right)^2 \\ & - \frac{12}{a} - \frac{3}{2} \frac{a}{b} - 2\frac{a}{b^2} + \frac{1}{a-1}\left(\frac{1}{2} + \frac{a}{b}\right) + G_0(a) \left[\frac{a^2}{b} + \frac{b}{a^2} + \frac{a}{b} + a\right. \\ & \left. + \frac{1}{2} b + \frac{2}{a} - \frac{3}{b} - 1\right]. \end{aligned} \qquad (7)$$

With this

$$4 \operatorname{sh}^2 y = -(a + b), \quad h(y) = y^{-1} \int_0^y u \operatorname{cth} u \, du,$$

$$G_0(a) = -2a^{-1} \int_{1-a}^{1} \ln(1 - u) \frac{du}{u};$$

$$m^2 k = 2p_1 q_1 = 2p_2 q_2; \quad m^2 \tau = 2p_1 q_2 = 2p_2 q_1,$$

where p_1 and p_2 are the electron and positron momenta, and q_1 and q_2 the photon momenta.

The quantity λ, appearing in $U^{(1)}$, represents the photon "mass". For $\lambda \to 0$ the radiative correction to W_0 diverges logarithmically (the infrared catastrophe). We must calculate W_{Free} for a relative particle velocity of zero. As can be seen from Eqs. (5) and (6), the infrared divergence disappears in that case. Actually if $p_1 \to p_2$, then $k \to \tau \to 0$. Therefore, the quantity y and with it the entire coefficient of $\ln \lambda$ in Eq. (6) goes to zero.

We finally obtain (in the center-of-mass system) the following expression for the probability of two photon annihilation of parapositronium with account of the first radiative corrections:

$$W = W_0 \left[1 - \frac{e^2}{\hbar c}\left(\frac{4}{3\pi} - \frac{\pi}{8} - \frac{2}{9\pi} \ln 2 - \frac{3}{2\pi} G_0(-2)\right)\right]$$

$$= \frac{1}{2}\left(\frac{e^2}{\hbar c}\right)^5 \frac{mc^2}{\hbar}\left[1 + 0.25 \frac{e^2}{\hbar c}\right].$$

In conclusion, we thank A. D. Galanin for discussion of the results.

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*We use the following summation convention: $ab = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3$.

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Energy Dependence of the Fission Width

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At excitation energies of the compound nucleus lying below the fission threshold E_f, the variation of the fission width with energy is determined mainly by the barrier factor. Near threshold the barrier factor has the form $e^{-\Delta E/\epsilon}$, where $\Delta E = E_f - E$. Theoretical estimates[1] give $\epsilon \sim 100$ kev; experimental data agree with this value. Thus, right up to $E = E_f$ the fission width Γ_f is a rapidly increasing function of energy. For $E = E_f$ the tunnel effect largely ceases, and in the region $E > E_f$ the behavior of the fission width is determined by other factors. The statistical theory of the fission width allows one — at least qualitatively— to analyze the dependence of $\Gamma_f(E)$ in this region. We[2] obtained previously the following estimate*:

$$\Gamma_f \sim (\hbar\omega / 2\pi)\, N^*(E - E_f) / N^*(E). \qquad (1)$$

Here ω is the frequency of the vibrations of the nuclear shape which are related to the fission, N^* is the number of levels not connected with the fission degrees of freedom. By $N^*(E)$ the number of levels with excitation energy less than E is indicated.

As shown by Landau[4] the density of levels with a given angular momentum goes basically as e^S, where $S = S(E)$ is the entropy. Therefore, the behavior of the denominator $N^*(N)$ is determined by the factor e^S, that is, it increases monotonically. As concerns the numerator $N^*(E - E_f)$, in the immediate region of the fission threshold it is impossible to apply the exponential factor to it. The formula e^S is valid for the number of levels only if the total number of levels is large and the distance between them small compared with the excitation energy. Near the threshold the number of levels $N^*(E - E_f)$ is the order of unity and the excitation energy $E - E_f$ cannot be considered large in comparison to the distance between levels with the same angular momentum.

Near threshold, the function $N^*(E - E_f)$ has a step-like character. Immediately after the point $E = E_f$ the interval $E - E_f$ contains only one level with the angular momentum considered, i.e., $N^* (E - E_f = 1$. In this region the fission width falls according to the law e^{-S}. Upon further increasing the energy, a second level with the angular momentum considered falls into the interval $E - E_f$, upon which the magnitude of $N^*(E - E_f)$ increases suddenly by a factor of 2, and the fission width also increases suddenly by a factor of 2. Then, there follows again a region in which the fission width drops according to the law e^{-S} until a third level occurs in the interval $E - E_f$, etc.

Thus, the dependence of the fission width on energy near to threshold is not monotonic. Of course, the picture described above of the step-like behavior of the fission width is idealized. In fact, the true curve $\Gamma_f(E)$ is found to be smoothed out on account of the tunnel effect and because of participation in the formation of the compound nucleus of neutrons with different angular momentum. It is necessary to take into account the fact that the neutrons used in the experiment are possibly not monochromatic. All the same, there is no reason to think that these factors lead to a complete smoothing out of the non-monotonic behavior of the fission width. For small excitation energies the distance between levels of the same angular momentum can reach ~ 1 mev and the factors enumerated above are not sufficient, generally speaking, to smooth out completely such non-monotonic features of the behavior of the fission width which occur so far apart. As the excitation energy increases these non-monotonic features become more frequent and less pronounced and for $E - E_f$ of the order of several mev they are almost completely smoothed out.

The non-monotonic behavior of the energy-dependence of the fission width near threshold should say something about the behavior of the fission cross section in this region. In fact, for the fission cross section for neutrons in the range $E_n \sim 1$ mev and higher we have

$$\sigma_f \sim \sigma_c \Gamma_f / \Gamma, \qquad (2)$$

where σ_c is the cross section for formation of the compound nucleus and Γ is the total width. In the region of interest the widths Γ_f and $\Gamma - \Gamma_f$ are apparently of the same order of magnitude and, consequently, the non-monotonic behavior of the function $\Gamma_f(E)$ leads to a non-monotonic behavior in the fission cross section $\sigma_f(E)$. For nuclei for which the fission by neutrons has a threshold character this circumstance is confirmed experimentally.[5]

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*The first estimate of the fission width was given by Bohr and Wheeler[3] who, starting from classical considerations, obtained $\Gamma_f \sim (D/2)\, N^* (E-E_f)$. As indicated in the previous note,[2] for $T \gg \hbar\omega$ the expression, Eq. (1), goes over into the formula of Bohr and Wheeler.

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59

Note on the Theorem of Pauli on the Relation of Spin and Statistics

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THE proof of Pauli[1] is based on a consideration of the irreducible representations of tensor and spinor quantities under transformations of the Lorentz group with determinant equal to unity. Schwinger[2] noted the connection of this theorem with the transformation of quantities under time reversal. We here give proof of the theorem of Pauli which shows that it is sufficient to restrict consideration to the transformation of quantities under inversion of all four coordinate axes (general inversion I) and which emphasizes the close connection of Schwinger's idea with the ideas of Pauli.

Under the transformation of general inversion I, we have for an arbitrary vector A

$$IA = -A, \tag{1}$$

from which it follows that under general inversion, tensors of even rank T_{2n+1} change sign. We say that a quantity belongs to the (+) class if it does not change under inversion I, and belongs to the (–) class if it changes sign. The operation of complex conjugation obviously does not change the class of tensor quantities.

We now consider the transformation of spinors under general inversion. Without restricting the generality, we can consider here and below only spinors of the first rank. As is well-known, upon reflection relative to a two-dimensional plane with normal vector a_k, the spinor U transforms according to the law*

$$U' = \hat{a}U, \quad \overline{U}' = \overline{U}\hat{a}, \tag{2}$$
$$\hat{a} = ia_k\gamma_5\gamma_k, \qquad \overline{U} = U^*\gamma_4.$$

We use the notation $\hat{a}^2 = a_k a_k$. We speak about space reflections if $a_k a_k = 1$, and about time reflections if $a_k a_k = -1$. It is easy to show that the bi-linear quantities composed of U and $\overline{U}$, which behave as tensors under spatial reflections, are pseudo-tensors under time reflections. Thus, for example, the scalar $(\overline{U}U)$ goes into $(\overline{U}\hat{a}^2 U)$, and the vector $(\overline{U}\gamma_k U)$ goes into

$$(\overline{U}\hat{a}\gamma_k\hat{a}U) = \hat{a}^2(\overline{U}\gamma_k U) - 2a_k a_i(\overline{U}\gamma_i U).$$

It is possible, however, so to change the definition of the laws of reflection of spinors that the bilinear tensors which are constructed from them behave as tensors not only for space reflections, but also for time reflections. This can be achieved if, in extending the concept of complex conjugation, two quantities, U^* and $-U^*$, are put into correspondence with each spinor U. For this we introduce, in analogy with the theory of functions of a complex variable, a "two sheet" space of spinors, where we arrange that the transformations which do not change the sign of the time, leave the spinor on the same sheet, and the transformations which change the sign of the time carry the spinor to the second sheet. The conjugate spinor U^* is

$$\begin{aligned} \overset{*}{U} &= U^* && \text{on the first sheet,} \\ \overset{*}{U} &= -U^* && \text{on the second sheet.} \end{aligned} \tag{3}$$

(The asterisk on the right indicates the usual complex conjugation.) The dual spinor $\overline{U}$ is defined by the equality

$$\bar{U} = \overset{*}{U}\gamma_4. \qquad (4)$$

If now, as above, we define the transformation of the spinor U by the equality (2), then the conjugate spinor transforms according to the law:

$$(\overset{*}{U})' = (\overset{*}{\hat{a}U}) = \overset{*}{U}\hat{a}^{+} \qquad (\text{ if } a_k \text{ is a space-like vector}),$$

$$(\overset{*}{U})' = (\overset{*}{\hat{a}U}) = -\overset{*}{U}\,\hat{a}^{+} \quad (\text{if } a_k \text{ is a time-like vector}). \qquad (5)$$

With this extension of the concept of conjugate spinor the bi-linear tensors constructed from U and $\bar{U}$ behave in the same way under both space and time reflections.

The transformation of inversion is a reflection of all four coordinate axes. Consequently, the matrix of inversion I in our representation is $i\gamma_5$:

$$IU = i\gamma_5 U. \qquad (6)$$

The conjugate spinor transforms according to the law

$$I\overset{*}{U} = (\overset{*}{IU}). \qquad (7)$$

Suppose, for definiteness, that the spinor U lies on the first sheet. Then, according to (3) and (3′) the formula (7) becomes

$$IU^{*} = -(IU)^{*} = iU^{*}\gamma_5. \qquad (7')$$

Since $(i\gamma_5)^2 = -1$, the eigen spinors of the inversion operator can belong to two classes: to the class $(+)$ if, upon inversion, they are multiplied by $+i$, and to the class $(-)$ if they are multiplied by $-i$. The formulas (6) and (7′) show that the complex conjugate spinors fall into the same class.

We turn now to consideration of the commutators for tensor and spinor quantities. Subjecting the known expression for the commutator of tensor quantities $A(x)$ to the inversion I

$$[A(x'),\ A^{*}(x'')]_{\pm} = P_n(\partial/\partial x)\,\Delta(x' - x''), \qquad (8)$$

taking into account that $\Delta(-x) = -\Delta(x)$ and that the quantities A and A^{*} belong to the same class, we obtain the relation

$$P_n(-\partial/\partial x) = P_n(\partial/\partial x). \qquad (9)$$

Consequently, $P_n(\partial/\partial x)$ is an even polynomial.

Considering the commutators for spinors, and taking into account the fact that U and U^{*} belong to the same class, we come to the conclusion

$$[U(x'),\ U^{*}(x'')]_{\pm} = P_{2n+1}(\partial/\partial x)\,\Delta(x' - x''), \qquad (10)$$

where $P_{2n+1}(\partial/\partial x)$ is an odd polynomial.

We will suppose that A and U are expanded in plane waves. Since the equation connecting the field components should be invariant under the transformation of general inversion, they should have the form

$$\Sigma k A^{(+)} = \Sigma A^{(-)}; \qquad \Sigma k A^{(-)} = \Sigma A^{(+)};$$

$$\Sigma k U^{(+)} = \Sigma U^{(-)}; \qquad \Sigma k U^{(-)} = \Sigma U^{(+)} \qquad (11)$$

(we use Pauli's notation).

From these equations it follows that—together with the solutions of Eq. (11) $A^{(+)}$, $A^{(-)}$ ($U^{(+)}$, $U^{(-)}$), corresponding to the value k – there exist $A^{(+)}$, $-A^{(-)}$ ($U^{(+)}$, $-U^{(-)}$), corresponding to the value $-k$.

We construct the tensors T_{2n} and T_{2n+1} of even and odd rank, respectively, depending on k and depending quadratically on $A^{(\pm)}$, $A^{*(\pm)}$ or $U^{(\pm)}$, $U^{*(\pm)}$. We require that the tensors T_{2n} and T_{2n+1} are gauge-invariant and that they behave in the proper way under general inversion. Following Pauli's notation, we have, in the case of spinor quantities,

$$T_{2n} = U^{(+)}U^{*(-)} + U^{*(+)}U^{(-)} + k\,(U^{(+)}U^{*(+)} + U^{(-)}U^{*(-)});$$

$$T_{2n+1} = k\,(U^{(+)}U^{*(-)} + U^{*(+)}U^{(-)}) + U^{(+)}U^{*(+)} + U^{(-)}U^{*(-)}. \qquad (12)$$

We show that the quantity $T_{2n}(U)$ cannot be positive. In fact, let it be positive for some $U(k) = U^{(+)}(k) + U^{(-)}(k)$. There exists, however, also a solution of Eq. (11) for which $\tilde{U}(-k) = U^{(+)}(k) - U^{(-)}(k)$. For this solution T_{2n} changes sign with the substitution of $-k$ for k.

Turning now to the quantities $T_{2n}(A)$, $T_{2n+1}(A)$ composed of tensor quantities, we have

$$T_{2n}(A) = A^{(+)}A^{*(+)} + A^{(-)}A^{*(-)} + k\,(A^{(+)}A^{*(-)} + A^{(-)}A^{*(+)});$$

$$T_{2n+1}(A) = k\,(A^{(+)}A^{*(+)} + A^{(-)}A^{*(-)}) + A^{(+)}A^{*(-)} + A^{(-)}A^{*(+)}.$$

Repeating the same considerations, we find that the quantity $T_{2n+1}(A)$ cannot be positive.

Considering only general inversion, we have shown that in the case of a spinor field the energy density cannot be positive definite, and in the case of a tensor field the energy density cannot be positive definite. From these premises and from the expressions, Eqs. (8) and (10), Pauli constructed the proof of his theorem.

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60

A Possible Application of Cyclotron Resonance to Mass Spectrometry

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CYCLOTRON resonance, which was theoretically predicted by Dorfman[1] and has been observed experimentally in semiconductors,[2] consists of absorption of microwave power by an assemblage of electrons or holes upon which a magnetic field H is acting. The absorption takes place at a frequency

$$\nu = eH/2\pi mc, \tag{1}$$

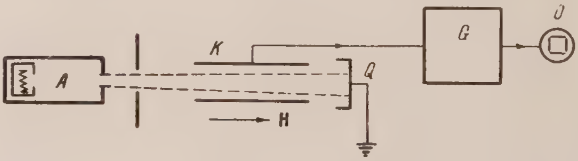
equal to the frequency of rotation of the carriers in the field H and is due to electric dipole transitions[3] whose possibility is controlled by the motion of the charges in quantum orbits. The electric dipole transitions are induced by the electric vector E_ν of the microwave field; the transition probability is a maximum when E_ν is perpendicular to H.

In comparison with paramagnetic resonance, cyclotron resonance absorption is distinguished by its remarkably large intensity. The electric dipole transition probability is approximately 10^{12} times larger than the probability of the magnetic dipole transitions in paramagnetic resonance.[4] Consequently the observation of cyclotron resonance requires a carrier concentration 10^{12} times smaller than is necessary for the observation of paramagnetic resonance. In view of the sensitivity of modern radiospectroscopic techniques it should be possible[4] to observe cyclotron resonance in a gas of free electrons at about 10^4 electrons per cm^3. Cyclotron resonance absorption must occur also in the space charge formed by ions of this or that substance; the relative magnitude of this absorption[3] must be 10^3 to 10^4 times less than for an electron gas under the same conditions of concentration and supplied *rf* power.

The condition that the effect be observable is that $\omega\tau > 1$. It follows from the requirement that the carrier be able to execute at least one complete rotation in the time between successive collisions.

Cyclotron resonance may find application in gas analysis, in studying the properties of plasma (e.g., in measuring ion mobilities and collision times), and in mass spectrometry.

The principle of operation of a possible mass spectrometer is similar to that of the *rf* mass spectrometer propsed by Hipple *et al.*[6] and consists of measuring the cyclotron frequencies of ions of the various masses and subsequently computing m by formula (1). A schematic diagram of the apparatus is shown in Fig. 1. The ion source A sends an unfocused ion beam of low energy into the condenser K which forms part of the resonant circuit of the high-frequency tunable oscillator G. In the field H, parallel to the faces of the condenser, the ions move along helical paths, whose projections upon a plane perpendicular to H are circles of radius $\rho = mcv_\perp / eH$, where $v_\perp$ is the projection of the ion velocity upon this same plane. Even if an ion executes only a single rotation during its flight through the condenser absorption of high-frequency power must be observable at the frequency (1). By modulating the field or the oscillator frequency it is possible by the usual means to display the resonance curve on an oscilloscope or to register it with a recorder. After



Schematic diagram of apparatus.

traversing the condenser the ions are discharged by the collector Q.

The sensitivity of the instrument is determined from the ratio of the number N of ions per cm^3 and the minimum density n required for detecting absorption, i.e., $N/n \approx N/10^4$. Thus the sensitivity will be the higher, the larger the ion current and the lower the ion energy. It is not difficult to see that in principle it is possible to raise this quantity to values of the order of 10^6 to 10^7 ; thus the possibility is disclosed of detecting isotopes having a natural abundance of a part in ten million.

The dispersion of the instrument is linear and according to formula (1) is obtained from the relation

$$D = (eH/2\pi mc)\,(1/100m) \qquad (2)$$

for a 1 percent change of mass. For elements of the center of the periodic table D will be about 10^4cps.

The resolving power is given by $R = m^*/\Delta m = \nu/\Delta\nu$, where $\Delta\nu$ is the error in the frequency measurement. Since $\Delta\nu = 10^{-5}\,\nu$ is easily attained, it is possible to hope for the creation of an instrument with a resolving power of about 10^5 and higher.

Thus experiments with cyclotron resonance on ion beams may prove to be useful in measuring nuclear masses. Such an experiment is easier to carry out for light elements.

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61

Scattering Lengths of Slow Neutrons on Deuterons

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THE scattering of slow neutrons on free deuterons is completely specified by the two scattering lengths a_4 and a_2 that are associated with the two possible spin states of the system composed of two neutrons and one proton. From the experimental values of the total and coherent scattering cross sections for neutrons on deuterons it is possible to establish[1,2] that a_4 and a_2 are given by either α) $a_4 = 6.2 \times 10^{-13}$ cm and $a_2 = 0.8 \times 10^{-13}$ cm or β) $a_4 = 2.4 \times 10^{-13}$ cm and $a_2 = 8.3 \times 10^{-13}$ cm. Since the choice between these two possibilities cannot be made without experiments with polarized particles, it has to be based on theory. The theoretical solution of the problem of scattering in a three-body system has been carried out only under considerable simplifications. According to the theoretical work of some[3,4] one had to give a preference to the values β because according to theory $a_2 > a_4$. Others[5,6] obtained the opposite inequality $a_2 < a_4$, and one had to consider the values α as being the correct ones.

In the present note qualitative considerations in favor of the values α are pointed out.

The potential energy of the system composed of two neutrons and one proton depends only upon three "internal" coordinates, for which one may choose the distances ξ and η of the two neutrons from the proton and the cosine μ of the angle between them. The wave function of the S-states of such a system will depend only on the internal variables ξ, η, and μ and on the spin variables σ_1, σ_2 and σ_3 of all three particles. In particular, the wave function of the quartet state may be represented in the form

$$\Psi^{(s)}_{3/2\,M} = \varphi_{3/2}(\xi, \eta, \mu)\,\chi_{3/2\,M}(\sigma_1, \sigma_2, \sigma_3), \qquad M = \pm 1/2, \pm 3/2,$$

where $\chi_{3/2\,M}$ is the spin wave function symmetric in an interchange of the spin coordinates of any pair of particles. Because of the identity of the two neutrons the complete wave function $\Psi_{3/2\,M}$ must be antisymmetric in a single interchange of the space and spin coordinates of the neutrons. Since $\chi_{3/2\,M}$ is symmetric, the space wave function must be antisymmetric in an interchange of the space coordinates of the neutrons. The transformation $\xi \to \eta$, $\eta \to \xi$, $\mu \to \mu$ corresponds to an interchange of the neutrons. Therefore the antisymmetry of the function $\varphi_{3/2}(\xi, \eta, \mu)$ is expressed by the simple equation

$$\varphi_{3/2}(\xi, \eta, \mu) = -\varphi_{3/2}(\eta, \xi, \mu). \qquad (1)$$

From (1) it follows immediately that $\varphi_{3/2} = 0$ for $\xi = \eta$. Thus, in the quartet state, events in which the neutron being scattered falls inside the deuteron are impossible. In other words, in the quartet state the *n–d* interaction is described by some effective potential of repulsion whose range of influence coincides with the dimensions of the deuteron. In this connection, the scattering length a_4 must be positive and exceed the "radius" of the deuteron, that is $a_4 \geq 4.3 \times 10^{-13}$ cm. Of the two possible pairs α and β of the experimental values for the scattering lengths only the values α satisfy the condition $a_4 \geq 4.3 \times 10^{-13}$ cm. Hence it is necessary to take $a_4 = 6.2 \times 10^{-13}$ cm and $a_2 = 0.8 \times 10^{-13}$ cm.

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Contribution to the Theory of π-Meson Disintegration

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LET us consider the problem of the decay of the π-meson according to the scheme

$$\pi \to \mu + \gamma + \nu,$$

assuming that the μ-meson has an anomalous magnetic moment $\mu_a = \mu' + e/2M$, where M is the μ-meson mass. We consider the meson interaction as scalar (a pseudoscalar interaction yields the same results[1]):

$$H_{\pi,\,\mu\nu} = g(\bar{\varphi}_\nu \varphi_\mu)\Psi_\pi + \text{compl. conj.}$$

The interaction of a μ-meson with a γ-ray is given by the expression

$$H_{\mu,\,\gamma} = -ie\hat{A} - {}^1\!/_2\, i\mu' \gamma_i \gamma_k F_{ik},$$

$$F_{ik} = \partial A_k / \partial x_i - \partial A_i / \partial x_k.$$

The matrix element of the process is given by

$$M = \frac{2\pi e g}{\sqrt{E_\pi |k|}}\, \bar{u}_\mu \times \left[\hat{e} - \frac{i\mu'}{2e}(\hat{k}\hat{e} - \hat{e}\hat{k})\right](i\hat{p} + i\hat{k} - M)^{-1} u_\nu$$

where u_μ, u_ν are unitary bispinors of the wave functions of the μ-meson, and of the neutrino; $k = (\mathbf{k}, |k|)$ is the 4-momentum of the photon; $p = (\mathbf{p}, M)$ is the 4-momentum of the μ-meson and e is the unit polarization vector of the photon.

Averaging over polarizations and spins, we get the decay probability

$$\begin{aligned} dw = \frac{1}{16\pi^3}\frac{e^2 g^2}{E_\pi E E_\nu k}\Big\{\Big[&-2(\mathbf{pk} - Ek) \\ &\times(-\mathbf{pk} - E_\pi k + Ek) - \frac{[\mathbf{pk}]}{k^2}(E_\pi^2 - [M - m]^2) \\ &+ 8\Big[2\left(\frac{\mu'}{2e}\right)^2(\mathbf{pk} - Ek)^2(-\mathbf{pk} + Ek) \\ &+ 2\left(\frac{\mu'}{2e}\right)(\mathbf{pk} - Ek)^2(M^2 - EE_\pi - mM) \\ &+ \left(-\frac{\mu'}{2e}\right)(\mathbf{pk} - Ek)^2(m - M) \\ &+ \left(-\frac{\mu'}{2e}\right)(\mathbf{pk} - Ek)(-ME_\pi k)\Big]\Big\}\frac{d\mathbf{p}\, d\mathbf{k}}{(Ek - \mathbf{pk})^2}, \end{aligned}$$

where m is the mass of the neutrino and E_π is the energy of the decaying meson.

Integrating over the directions of the photon, we get the probability of a decay with emission of a μ-meson with momentum p (we take $m = 0$):

$$dw=\frac{e^2g^2}{2\pi}\frac{p^2dp}{EE_\pi}\left\{\frac{(E_\pi-E)^2-p^2}{pE_\pi}\right.$$

$$\times\ln\frac{(E+p)(E_\pi-E+p)}{(E-p)(E_\pi-E-p)}+\frac{4}{p}\frac{E_\pi^2-M^2}{(E_\pi-E)^2-p^2}\left(-2p\right.$$

$$\left.+E\ln\frac{E+p}{E-p}\right)-4\left(-\frac{\mu'}{2e}\right)$$

$$\times M\frac{(E_\pi-E)^2-p^2}{pE_\pi}\ln\frac{(E+p)(E_\pi-E+p)}{(E-p)(E_\pi-E-p)}$$

$$\left.-8\left(\frac{\mu'}{2e}\right)^2M^2\left(1-\frac{EE_\pi}{M^2}\right)\right\}.$$

In the non-relativistic case, $p \gg M, E_\pi$; $E = M + p^2/2M$. Assuming [2] that the mean free path R of the μ-meson is proportional to p^4 we get for the number of μ-mesons with mean free paths less than R

$$w=\int_0^{p=p_0(R/R_0)^{1/4}}dw=[1+\tau]\,w_{\mu+\nu+\gamma}$$

$$+\frac{e^2g^2}{2\pi}\frac{M^2}{E_\pi}\int_0^{(p_0/M)(R/R_0)^{1/4}}\left[\frac{\tau^2}{2}\left(\frac{E_\pi}{M}-1\right)x^2\right.$$

$$\left.+\frac{8\tau}{3}\left(-\frac{M}{E_\pi}+\frac{E_\pi}{M}\right)\frac{x^4}{(E_\pi/M-1)^2-x^2}\right]dx.$$

Here p_0 and R_0 are the momentum and the mean free path of the meson in the decay $\pi\to\mu+\nu$; $\tau=\mu'/(e/2M)$; $w_{\mu+\nu+\gamma}$ is the decay probability for $\tau=0$ derived by Ioffe and Rudik[1]; $w_{\mu+\nu}=(g^2/2)(1-M^2/E_\pi^2)\,P_0$ is the probability of the decay $\pi\to\mu+\nu$.

The comparison with the results of Ioffe and Rudik[1] shows that the μ-meson having an anomalous magnetic moment can lead to an increase of the number of mesons especially of those with short mean free paths. Similar results should be expected in the case of mesons with spin greater than ½.

I wish to thank B. L. Ioffe for the suggestion of this problem and its discussion.

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Production of π-Meson Pairs on Nuclei by High Energy γ-Quanta

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THE production of π-meson pairs on nuclei by high energy γ-quanta has been discussed by Pomeranchuk.[1,2] In the case of high energy γ-quanta, only small angles between the momenta of the π-mesons and the γ-quanta are important. The range of the process is found to be greater than the dimensions of the nucleus. Therefore, the knowledge of the wave function outside the nucleus is sufficient to determine the cross section of the process. In Refs. 1 and 2, the wave function was taken as a plane wave plus a wave scattered by a perfectly black sphere of radius R (radius of the nucleus). In this paper we take into account the influence of the Coulomb interaction between the π-mesons and the charge of the nucleus on the pair formation.

The matrix element of the process of formation of a π^+, π^- – pair is given by

$$M=-ie\sqrt{\frac{2\pi}{\omega}}\int[\psi_+^*(\mathbf{j}\cdot\nabla)\psi_-^*-\psi_-^*(\mathbf{j}\nabla)\psi_+^*]\,e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r},\quad(1)$$

where $\mathbf{k}$, ω and $\mathbf{j}$ are the wave vector, the frequency and the polarization of the incoming quantum. The wave functions ψ_+ and ψ_- of the created mesons are the sums of plane and converging waves:

$$\psi_+=\frac{1}{\sqrt{2E_+}}\left\{e^{i\mathbf{p}_+\mathbf{r}}\right.\qquad(2)$$

$$\left.+\frac{p_+}{2\pi i}\int\frac{\exp\{-ip_+|\mathbf{r}-\rho|\}}{|\mathbf{r}-\rho|}\{1-\Omega^*(\rho)\}\,d\rho\right\},$$

where ρ is the radius in a plane orthogonal to the momentum $\mathbf{p}_+$ of the created meson and passing through the center of the nucleus:

$$\Omega(\rho)=\begin{cases}0 & \rho\leqslant R,\\ e^{2i\eta_+(\rho)}, & \rho>R;\end{cases}$$

$\gamma_+(\rho)\approx n_+\log p+\rho$ is the Coulomb scattering phase, with $n_+=ze^2E_+/p_+$ and E_+ is the energy of the π^+-meson. We break the wave function into three parts:

$$\psi_+ = \frac{1}{\sqrt{2E_+}}\{e^{i\mathbf{p}_+\mathbf{r}} + \Phi_+ + \Phi'_+\}; \tag{2'}$$

$$\Phi_+ = \frac{p_+}{2\pi i}\int\limits_{\rho\leqslant R}\frac{\exp\{-ip_+|\mathbf{r}-\boldsymbol{\rho}|\}}{|\mathbf{r}-\boldsymbol{\rho}|}\,d\boldsymbol{\rho};$$

$$\Phi'_+ = \frac{p_+}{2\pi i}\int\limits_{\rho>R}\frac{\exp\{-ip_+|\mathbf{r}-\boldsymbol{\rho}|\}}{|\mathbf{r}-\boldsymbol{\rho}|}\{1-e^{-2in_+(\rho)}\}\,d\boldsymbol{\rho};$$

similarly for π^- :

$$\psi_- = (2E_-)^{1/2}\{e^{i\mathbf{p}_-\mathbf{r}} + \Phi_- + \Phi'_-\}; \tag{3}$$

$$n_- = -Ze^2E_-/p_-.$$

In the evaluation of the matrix element (1), let us note that the integrals containing the products of the plane waves involved in ψ_+ and ψ_- vanish because of the conservation laws; all the integrals containing the products of non-overlapping wave functions (ψ_+ and ϕ_- , ψ_- and ϕ_+, ϕ_+ and ϕ'_- and finally ϕ_- and ϕ'_+ do also vanish. We get

$$M = -ie\sqrt{\frac{\pi}{2\omega E_+ E_-}} \tag{4}$$

$$\times\int\{(\Phi^*_+ + \Phi'^*_+)\,\mathbf{j}\cdot\nabla\, e^{-i\mathbf{p}\cdot\mathbf{r}} + e^{-i\mathbf{p}_+\cdot\mathbf{r}}\,\mathbf{j}\cdot\nabla)\,\Phi'^*$$

$$-(\Phi^*_- + \Phi'^*_-)\,\mathbf{j}\cdot\nabla\, e^{-i\mathbf{p}_+\mathbf{r}} - e^{-i\mathbf{p}_-\cdot\mathbf{r}}\,\mathbf{j}\cdot\nabla)\,\Phi'^*_+\}\,e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r}.$$

In the case of relativistic energies (E_+ and $E_- >> \mu$) and of small angles between $\mathbf{p}_+$, $\mathbf{p}_-$ and $\mathbf{k}$ (which is the only case where the diffraction treatment is adequate), and assuming that $n << 1$ we have:

$$M = ie\sqrt{8\pi^3 p_+ p_-/\omega^3} \tag{5}$$

$$\times\left[\frac{\mathbf{j}\xi}{1+\xi^2} - \frac{\mathbf{j}\eta}{1+\eta^2}\right]\frac{1}{2\mu^2}\frac{R}{q}J_1(2\mu Rq)$$

$$+ e\sqrt{8\pi^3 p_+p_-/\omega^3}\,\frac{n^2}{\mu^3 q^2}$$

$$\times\left\{\frac{\mathbf{j}\xi}{1+\xi^2}[J_0(2\mu Rq) + 2\mu RqJ_1(2\mu Rq)\ln p_-R]\right.$$

$$\left.+\frac{\mathbf{j}\eta}{1+\eta^2}[J_0(2\mu Rq) + 2\mu RqJ_1(2\mu Rq)\ln p_+R]\right\},$$

where $q = \frac{1}{2}(\xi + \eta)$ and ξ and η are determined by the relations

$$\mathbf{p}_+ = p_+(1-\xi^2\mu^2/2p_+^2)\frac{\mathbf{k}}{\omega} + \mu\boldsymbol{\xi},\quad \mathbf{k}\cdot\boldsymbol{\xi} = 0,$$

$$\mathbf{p}_- = p_-(1-\eta^2\mu^2/2p_-^2)\frac{\mathbf{k}}{\omega} + \mu\boldsymbol{\eta},\quad \mathbf{k}\cdot\boldsymbol{\eta} = 0.$$

The differential cross section for pair formation is equal to

$$d\sigma_j(E_+,\ \xi,\ \eta) \tag{6}$$

$$= 2\pi\,|M^2|\,|F|^2\,\mu^4\,(2\pi)^{-6}\,d\xi\,d\eta\,dE_+,$$

where F is a form factor taking into account the finite size of the particles and their interaction between themselves. Substituting (5) into (6), and averaging over polarization of the quanta, we get:

$$d\sigma = \frac{e^2p_+p_-}{32\pi^2\omega^3}\left\{\frac{R^2J_1^2(2\mu Rq)}{q^2}\left[\frac{\xi}{1+\xi^2} - \frac{\eta}{1+\eta^2}\right]^2\right. \tag{7}$$

$$+\frac{4n^2}{\mu^2q^4}\left[\frac{\xi}{1+\xi^2}(J_0(2\mu Rq) + 2\mu RqJ_1(2\mu Rq)\ln p_-R)\right.$$

$$+\frac{\eta}{1+\eta^2}(J_0(2\mu Rq)$$

$$\left.\left.+2\mu RqJ_1(2\mu Rq)\ln p_+R)\right]^2\right\}|F|^2\,dE_+\,d\xi\,d\eta.$$

In (5) the most important lengths are the $r_{\text{eff.}} \sim 2p_+p_-/\omega\mu^2$; the effective value of q is therefore approximately equal to $1/\mu r_{\text{eff.}}$ Furthermore $r_{\text{eff.}} >> R$; hence $2\mu Rq_{\text{eff.}} << 1$ and

$$d\sigma = \frac{e^2E_+(\omega - E_+)}{32\pi^2\omega^3} \tag{8}$$

$$\times\left\{\frac{R^2J_1^2(2\mu Rq)}{q^2}\left(\frac{\xi}{1+\xi^2} - \frac{\eta}{1+\eta^2}\right)^2\right.$$

$$\left.+\frac{4n^2J_0^2(2\mu Rq)}{\mu^2q^4}\left(\frac{\xi}{1+\xi^2} + \frac{\eta}{1+\eta^2}\right)^2\right\}|F|^2\,dE_+\,d\xi\,d\eta.$$

The first term in the { } brackets corresponds to the diffractional $\pi^+\pi^-$ pair formation, without excitation of the nucleus [2] ; the second term takes into account the pair formation produced by the Coulomb interaction. For $n << 1$, the interference between the diffractional and Coulomb pair formation does not appear. Setting $F = 1$ and integrating (8) over ξ and η , we get

$$d\sigma = d\sigma^d + d\sigma^c; \tag{9}$$

$$d\sigma^d = \frac{e^2R^2}{2}\varepsilon(1-\varepsilon)\,d\varepsilon\left[\ln\frac{1+\xi^2_{\max}}{e} + \frac{1}{1+\xi^2_{\max}}\right];$$

$$d\sigma^c = \frac{2e^2n^2}{\mu^2}\varepsilon(1-\varepsilon)\,d\varepsilon\int\limits_{q_{\min}}^{\infty}\frac{J_0^2(2\mu Rq)}{q^3}\varphi(q)\,dq;\ \varepsilon = \frac{E_+}{\omega},\ q_{\min}$$

$$= \frac{\mu}{4\varepsilon(1-\varepsilon)\omega};\ \varphi(q) = \frac{2q^2+1}{q\sqrt{1+q^2}}\ln(q+\sqrt{1+q^2}) - 1;$$

the maximum value $\xi_{max} = \eta_{max}$ is determined by the properties of the π-particles interactions,[2] and q_{min} is determined from the conservation laws. Noting that the most important quantity in $d\sigma^c$ is $q << 1$ and setting $\varphi(q) \approx 4/3\, q^2$, we get a known result[3] for the cross section for pair formation in the Coulomb field:

$$d\sigma^c = (8e^2n^2/3\mu^2)\,\varepsilon(1-\varepsilon) \times d\varepsilon \ln[(2\varepsilon(1-\varepsilon)\,\omega/\mu^2R];\quad 2\mu R q_{min} \ll 1. \tag{10}$$

The integral cross sections are equal to

$$\sigma^d = \frac{e^2R^2}{12}\left[\ln\frac{1+\xi^2_{max}}{e} + \frac{1}{1+\xi^2_{max}}\right]; \tag{11}$$

$$\sigma^c = \frac{4}{9}\,\frac{e^2n^2}{\mu^2}\ln\frac{2\omega}{\mu^2R}.$$

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Invalidity of the Fermi-Dirac Distribution for Electrons of Semiconductor and Crystal Phosphor Impurity Centers

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AMONG the original assumptions in introducing the Fermi-Dirac distribution we have the following: 1) The electrons must not interact with each other, in particular, the energy of a given single electronic state must not depend on the distribution of electrons in the states ; 2) subject to the well known limitations, interaction of electrons is permitted with other subsystems (for example, if the state of this subsystem follows the electronic motion adiabatically). But it is necessary that the entropy of this subsystem not depend on the electronic distribution of the state.

It is simple to demonstrate that this assumption is not usually realized for electrons of semiconductor impurity centers. For example, if the donor is an atom of a monovalent metal, then the essential interaction of the electrons (other than the valence electrons) with one another and with the valence electrons, is still not a difficulty since the state of strongly bound electrons of the ion core follows the motion of the valence electrons adiabatically (and thus the motion of the conduction electrons). Therefore the ion core can be considered as the above-mentioned subsystem and the statistical distribution can be introduced only for the valence electrons of the donor and the conduction electrons. But the essential difficulty in obtaining a Fermi distribution is the following circumstance : there exist two equilibrium states of the valence electrons corresponding to the two possible orientations of their spin; hence, if one is occupied by an electron the energy level of the second state is raised considerably and even gets into the conduction band (this is consistent with the instability of the negative ions of the alkali elements, when introduced into crystals). Hence this violates the original assumption 1).

If the donor is an atom of a divalent element, in which the valence electrons have opposite spin orientations, then the state of the atom is nondegenerate. But after a single ionization of the donor, the state of the remaining electrons shows a twofold degeneracy corresponding to the two spin orientations of a single valence electron. Thus the degeee of degeneracy of the states of the subsystem consisting of ionic core is equal to 2^{N_1}, where N_1 is the number of singly ionized donors. In this case the entropy of the subsystem depends essentially on the electron distribution, i.e., in violation of assumption 2).

In consequence of the violation of the original assumptions 1) and 2), the Fermi-Dirac distribution, generally speaking, is not applicable to the donor electrons. Instead, we are led to employ the more general Gibbs distribution for a system with a variable number of particles, and we regard the donor as a system capable of losing electrons to the surrounding medium, and absorbing electrons from the medium. The probability that a donor will contain N electrons and be found in a quantum state n, is equal to

$$w_{nN} = \exp\{(\Omega + \mu N - E_{nN})/kT\}. \tag{1}$$

Here E_{nN} is the energy of all of the electrons of the donor (the electrons may interact with each other), Ω is the thermodynamic potential of the donor electrons, determined from the normalization condition $\Sigma_{n,N}\, w_{nN} = 1$ and equal to

$$\Omega = -kT \ln \sum_{n,N} \exp \left\{\frac{\mu N - E_{nN}}{kT}\right\}, \qquad (2)$$

β is the chemical potential of the electrons, which is the same for all electrons of the subsystem, which enter into the crystal. The average number of donor electrons is equal to

$$\bar{N} = \sum_{n,N} N w_{nN} \qquad (3)$$

$$= \sum_{n,N} N \exp \left\{\frac{\mu N - E_{nN}}{kT}\right\} \Big/ \sum_{n,N} \exp \left\{\frac{\mu N - E_{nN}}{kT}\right\}.$$

This formula is the generalization of the Fermi-Dirac distribution. For not too high temperatures, we can limit our consideration to singly ionized donors only. In this case, let the number of electrons in the ion core be equal to ν, and the number of electrons in neutral donors be $\nu + 1$. Then the Fermi "occupation number" of the donor electrons in our case corresponds to the number $N' = N - \nu$, which is equal to 1, if the donor is neutral, and to 0, if the donor is singly ionized. From Eq. (3) we obtain in this case (N takes on the values ν and $\nu + 1$ only):

$$\bar{N}' = \bar{N} - \nu \qquad (4)$$

$$= \left[1 + \sum_{n} \exp \left\{-\frac{E_{n\nu}}{kT}\right\} \Big/ \sum_{n} \exp \left\{\frac{\mu - E_{n\nu+1}}{kT}\right\}\right]^{-1}.$$

Here appear the partition functions of the ion core and the neutral donors. If we limit ourselves to the first term (low temperatures), which corresponds to the usual simplest popular variant of the theory of semiconductors, in which the interaction between the states of the ion core and the neutral donors is ignored, then we obtain from Eq. (4)

$$N' = [1 + \delta \exp \{(\varepsilon - \mu)/kT\}]^{-1}, \; \delta = f/g. \qquad (5)$$

Here the zero of energy is selected, usually so that the energy of the ion core in the ground state $E_{0\nu}$ is equal to zero, the energy of the electrons at the bottom of the conduction band equals zero, the energy of the neutral donors in the ground state is $E_{0\nu+1} = \epsilon$; f is the degree of degeneracy of the ground state of the ion fragment and g is the degree of degeneracy of the ground state of the neutral donor. Equation 5 differs from the usual Fermi-Dirac distribution by the factor δ, which cannot be eliminated by means of introducing the new value of μ, because it then loses its meaning as the chemical potential and does not conform to the chemical potential of the other electronic systems which make up the crystal.

If we use Eq. (5) we derive (by the usual method) the expression for the concentration of conduction electrons in semiconductors with a single type of donor and find

$$n_e = (G\delta n_d)^{1/2} (m^* kT/2\pi\hbar^2)^{3/4} e^{-w/2kT}, \qquad (6)$$

$$w = -\varepsilon > 0.$$

Here n_d is the concentration of donors, m^* is the effective mass of the conduction electrons, and G is the number of different quantum states of the conduction electrons with a given energy and quasi momentum. For example, if the conduction band is formed from atomic s-states, then $G = 2$, since there are two possible orientations of the spin of the electrons with a given energy and quasi-momentum.

Eq. (6) differs by the factor $\sqrt{\delta}$ from the analogous formula, deduced from the Fermi-Dirac statistics. Such an additional factor appears in the expression for the current in thermionic emission by semiconductors. An analogous correction appears in other equations in the theory of semiconductors, based on the statistics of electrons.

If the donor is an atom of a monovalent metal, then, as was explained above, $f = 1$, $g = 2$, $\delta = ½$. In the case when the donor is a bivalent element, then according to the above, $f = 2$, $g = 1$, $\delta = 2$. In trivalent donors the single p-electron can be in any one of the six p-states, so that after this electron leaves, the ion fragment is in the non-degenerate state. Hence $f = 1$, $g = 6$, $\delta = 1/6$. Strictly speaking, the six-fold p-levels of the donor atoms are slightly separated under the influence of the crystalline field, which does not possess spherical symmetry. This may be taken into account if we do not use Eq. (5) but the more exact, more general Eq. (4).

From the examples considered earlier, it is clear that not only Eq. (4), but even Eq. (5) leads to results different from the results from Fermi-Dirac statistics. However the latter, in the course of many years, has been widely applied to electrons of impurity centers, beginning first in the work of Wilson[1] and Fowler[2] and ending in the latest review of Schottky.[3] One should make the appropriate corrections to the formulae of the theory of semiconductors. In Ref. 4, the correct results are derived, since we do not use the Fermi-Dirac electron statistics but use the much more general Gibbs distribution.

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Application of the Lattice Model to Semiconductors of the Lead Sulphide Type

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THE application of the lattice model to solids was first pointed out by Coulson[1], who calculated (with the aid of this model) the band structure of the energy spectrum in metals with a cubic space lattice, and also in graphite and in boron nitride. In the present paper the lattice model is applied to semiconductors of the type PbX, where X = S, Se and Te. The specified semiconductors have cubic lattices of the NaCl type. The initial electron configurations in the isolated atoms are: $\mathrm{Pb}-p^2$, $X-p^4$. For each of the paired atoms of PbX in the crystal there are six p-electrons. We may consider two lattice models of PbX crystals:

a) the three-dimensional model, analogous to the three-dimensional lattice model, proposed by Coulson for metals;

b) it is possible to describe the p-electrons in the crystal as breaking up into the groups p_x, p_y and p_z, and the electrons of each group break up into subgroups, such that all electrons of one subgroup can move in a definite plane of the tetragonal lattice, formed by the atoms of Pb and X. Each of the subgroups is similar to a system of π-electrons in conjugate organic molecules.

In the first, as well as in the second model, we set the following for the potential V along the lattice: $V = V_0$ within the Pb atoms and $V = 0$ within the X atoms (V_0 is the difference between the electronegativities of X and Pb). For conjugate organic molecules containing heterocyclic atoms, Veselov and Rekasheva[2] proposed such a form for the potential.

In order to determine the size of the atoms of Pb and X in the crystal, we start out from the value of the ionic radii of Pb^{++} and X^{--} (according to Goldshmidt). The initial values of the ionic radii r_{Pb} and r_X are changed in proportion so that in each crystal the equation $r_{\mathrm{Pb}} + r_X = a$ holds (a is the distance between the atoms Pb and X).

We have the following relation for the binding energy of the electron and its quasi-momentum **k**:

In the first model

$$f(E) = {}^1/_9 (\cos ak_x + \cos ak_y + \cos ak_z)^2;$$

In the second model

$$f(E) = {}^1/_4 (\cos ak_x + \cos ak_y)^2.$$

The function $f(E)$ has the following form:

$$f(E) = (\cos \omega_1 r_X \operatorname{ch} \omega_2 r_{\mathrm{Pb}} - (\omega_1 / \omega_2) \sin \omega_1 r_X \operatorname{sh} \omega_2 r_{\mathrm{Pb}})$$

$$\times (\cos \omega_1 r_X \operatorname{ch} \omega_2 r_{\mathrm{Pb}} + (\omega_2 / \omega_1) \sin \omega_1 r_X \operatorname{sh} \omega_2 r_{\mathrm{Pb}}),$$

if $E < V_0$ and

$$f(E) = (\cos \omega_1 r_X \cos \omega_2 r_{\mathrm{Pb}}$$

$$- (\omega_1 / \omega_2) \sin \omega_1 r_X \sin \omega_2 r_{\mathrm{Pb}})$$

$$\times (\cos \omega_1 r_X \cos \omega_2 r_{\mathrm{Pb}} - (\omega_2 / \omega_1) \sin \omega_1 r_X \sin \omega_2 r_{\mathrm{Pb}}),$$

$$\omega_1 = \sqrt{2E}; \quad \omega_2 = \sqrt{2 \mid E - V \mid},$$

if $E > V_0$. The allowed energy values in the first and second models are determined from the equation

$$0 \leqslant f(E) \leqslant 1.$$

In this manner the three-dimensional and the plane model lead to the same band structure for the energy spectrum. At absolute zero, the first allowed band in the first as well as in the second model is completely filled with electrons, the remaining allowed zones are empty.

We calculated the width of the forbidden zone ΔE, the effective mass of the holes m_h^* and the effective mass of the electrons m_e^* (in units of the electronic mass) in the semiconductors under consideration:

	r_{Pb}, Å	r_X, Å	V_0, eV	ΔE, eV	$\Delta E/\Delta E^{(1)}_{exp}$	$\Delta E/\Delta E^{(2)}_{exp}$	m^*_h	m^*_e
PbS	1.28	1.68	2.50	1.48	1.42	1.06	2.03	1.02
PbSe	1.25	1.82	1.95	1.13	1.36	1.08	1.54	0.88
PbTe	1,22	1.95	1.39	0.79	1.27	0.88	1.07	0.70

There are presented in the Table the values of the effective mass corresponding to the second model. The values which correspond to the first model can be derived by multiplying them by 2.25. The experimental values of the widths of the forbidden zones $\Delta E^{(1)}_{exp}$ are taken from Ref. 3, $\Delta^{(2)}_{exp}$ from Ref. 4. The value of V_0 was taken from Ref. 5. Notice that the derived value of ΔE, and also the values of the effective mass, may change somewhat, owing to some uncertainty in the values of the parameters r_{Pb}, r_X and V_0. Nevertheless, we think that the results obtained confirm the applicability of the lattice model to semiconductors and the desirability of extending its applications in this direction.

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Photoproduction of π–Meson Pairs in Hydrogen and Deuterium Near Threshold and Isotopic Invariance

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THE cross section for photoproduction of π–meson pairs in hydrogen and deuterium is calculated by considering the interaction of an electromagnetic field with a meson-nucleon system in first order perturbation theory (it is well known that this does not presuppose a small interaction between mesons and nucleons). The interaction operator consists of two parts[1] $H = S + V_3$, where S is a scalar and V_3, the third component of a vector is isotopic spin space.

The isotopic spin of the initial state in the reaction

$$\gamma + p \rightarrow n' + \pi' + \pi'' \tag{1}$$

is equal to ½. The isotopic spin of the meson-nucleon system in the final state may be $t = ½$ and $t = 3/2$. The operator S yields transitions to states of the meson-nucleon system with isotopic spin $t = ½$, and the operator V_3 to states $t = ½$ and $t = 3/2$.

S_{Tt} V_{Tt} will be used to denote transition amplitudes to states of total isotopic spins $t = ½$ and 3/2 of the meson-nucleon system, and isotopic spin T of the two π–mesons system ($T = 0, 1, 2$). The differential cross section for photoproduction of two π–mesons then becomes

$$\sigma_1 (\gamma + p \rightarrow p + \pi^0 + \pi^0) \tag{2}$$

$$= \left| \frac{1}{\sqrt{3}} S_{0\,{}^1/_2} + \frac{1}{3} V_{0\,{}^1/_2} + \frac{2}{3}\sqrt{\frac{2}{5}}\, V_{2\,{}^3/_2} \right|^2 ;$$

$$\sigma_2 (\gamma + p \rightarrow p + \pi^+ + \pi^-)$$

$$= \left| \frac{1}{\sqrt{3}} S_{0\,{}^1/_2} + \frac{1}{3} V_{0\,{}^1/_2} - \frac{1}{3}\sqrt{\frac{2}{5}}\, V_{2\,{}^3/_2} \right.$$

$$\left. + \frac{1}{\sqrt{6}} S_{1\,{}^1/_2} + \frac{1}{3\sqrt{2}} V_{1\,{}^1/_2} - \frac{\sqrt{2}}{3} V_{1\,{}^3/_2} \right|^2 ; \tag{3}$$

$$\sigma_3(\gamma + p \rightarrow n + \pi^+ + \pi^0) \tag{4}$$

$$= \left| -\frac{1}{\sqrt{3}} S_{1\,1/2} - \frac{1}{3} V_{1\,1/2} - \frac{1}{3} V_{1\,3/2} + \frac{1}{\sqrt{5}} V_{2\,3/2} \right|^2 .$$

Near threshold, the isotopic spin T of the two π–mesons system can only be 0 or 2. Indeed, the wave functions for the states of isotopic spin $T = 0$ and 2 are symmetric with respect to charge exchange of the π–mesons coordinates (pions obey Bose statistics). The wave function for the state with isotopic spin $T = 1$ is antisymmetric with respect to charge exchange of the π–mesons and thus antisymmetric in the coordinates. Near threshold, π–mesons from reaction (1) are formed in the S state. Since the wave function for the final state must be symmetric with respect to permutations of the π–mesons, the production of two π–mesons with $T = 1$ is forbidden near threshold[1].

Substitution of S_{1t}, $V_{1t} = 0$ in Eqs. (2) – (4) yields the following inequalities among the differential cross sections for reactions near the threshold:

$$3(\sigma_1 + 2\sigma_2) \geqslant 4\sigma_3, \tag{5}$$

$$\sigma_1^{1/2} + \sigma_2^{1/2} \geqslant \sqrt{2}\,\sigma_3^{1/2} \geqslant |\sigma_1^{1/2} - \sigma_2^{1/2}|,$$

$$\sigma_1^{1/2} + \sqrt{2}\,\sigma_3^{1/2} \geqslant \sigma_2^{1/2} \geqslant |\sigma_1^{1/2} - \sqrt{2}\,\sigma_3^{1/2}|,$$

$$\sigma_2^{1/2} + \sqrt{2}\,\sigma_3^{1/2} \geqslant \sigma_1^{1/2} \geqslant |\sigma_2^{1/2} - \sqrt{2}\,\sigma_3^{1/2}|$$

and among total cross sections σ^t:

$$3(\sigma_1^t + \sigma_2^t) \geqslant 2\sigma_3^t,\ \sigma_1^{t\,1/2} \tag{6}$$

$$+ \frac{1}{\sqrt{2}}\sigma_2^{t\,1/2} \geqslant \sigma_3^{t\,1/2} \geqslant \left| \sigma_1^{t\,1/2} - \frac{1}{\sqrt{2}}\sigma_2^{t\,1/2} \right|,$$

$$\sigma_1^{t\,1/2} + \sigma_3^{t\,1/2} \geqslant \frac{1}{\sqrt{2}}\sigma_2^{t\,1/2} \geqslant |\sigma_1^{t\,1/2} - \sigma_3^{t\,1/2}|,$$

$$\sigma_3^{t\,1/2} + \frac{1}{\sqrt{2}}\sigma_2^{t\,1/2} \geqslant \sigma_1^{t\,1/2} \geqslant \left| \sigma_3^{t\,1/2} - \frac{1}{\sqrt{2}}\sigma_2^{t\,1/2} \right| .$$

Now let us consider the photoproduction of two π–mesons in deuterium, i.e., the reaction

$$\gamma + d \rightarrow n_1 + n_2 + \pi' + \pi'' . \tag{7}$$

The initial state has isotopic spin $t = 0$. The final state consists of a superposition of states with isotopic spins $t = 0$ and $t = 1$. The corresponding transition operators are S and V_3. S_{Tt_1} and V_{Tt_1} will denote transition amplitudes to states of isotopic spin T of the two π–mesons system, isotopic spin t of the two nucleons, and total spin $t = 0\,(S_{Tt_1})$ and $t = 1\,(V_{Tt_1})$. Various processes then give

$$\sigma_1(\gamma + d \rightarrow p + n + \pi^0 + \pi^0) \tag{8}$$

$$= \frac{1}{6}\left| S_{00} - V_{01} - \frac{2}{\sqrt{5}} V_{21} \right|^2 ;$$

$$\sigma_2(\gamma + d \rightarrow p + n + \pi^+ + \pi^-) \tag{9}$$

$$= \frac{1}{6}\left| -S_{00} + \frac{1}{\sqrt{2}} S_{11} + V_{01} + \sqrt{\frac{3}{2}} V_{10} - \frac{1}{\sqrt{5}} V_{21} \right|^2$$

$$\sigma_3(\gamma + d \rightarrow p + p + \pi^- + \pi^0) \tag{10}$$

$$= \left| \frac{1}{6} \right| S_{11} + \sqrt{\frac{3}{2}} V_{11} + \frac{3}{\sqrt{10}} V_{21} \right|^2 .$$

If the reaction takes place without deuteron breakup, T can only be 0 or 1; therefore,

$$\sigma(\gamma + d \rightarrow d + \pi^0 + \pi^0) = {}^1/_6\,|S_{00}|^2; \tag{11}$$

$$\sigma(\gamma + d \rightarrow d + \pi^+ + \pi^-) = {}^1/_6\,|-S_{00} + \sqrt{{}^3/_2}\,V_{10}|^2 .$$

Integration of the cross section for the reaction $\gamma + d \rightarrow d + \pi^+ + \pi^-$ eliminates interference terms and the total cross section σ^t becomes

$$2\sigma^t(\gamma + d \rightarrow d + \pi^0 + \pi^0) \leqslant \sigma^t(\gamma + d \rightarrow d + \pi^+ + \pi^-). \tag{12}$$

Near threshold, S_{1t_1} and $V_{1t_1} = 0$, and the following inequalities apply:

$$3(\sigma_1 + 2\sigma_2) \geqslant 4\sigma_3; \tag{13}$$

$$\sigma_1^{1/2} + \sigma_2^{1/2} \geqslant \sqrt{2}\,\sigma_3^{1/2} \geqslant |\sigma_1^{1/2} - \sigma_2^{1/2}|;$$

$$\sigma_1^{1/2} + \sqrt{2}\,\sigma_3^{1/2} \geqslant \sigma_2^{1/2} \geqslant |\sigma_1^{1/2} - \sqrt{2}\,\sigma_3^{1/2}|;$$

$$\sigma_2^{1/2} + \sqrt{2}\,\sigma_3^{1/2} \geqslant \sigma_1^{1/2} \geqslant |\sigma_2^{1/2} - \sqrt{2}\,\sigma_3^{1/2}| .$$

And among total cross sections σ^t:

$$3(\sigma_1^t+\sigma_2^t)\geqslant 4\sigma_3^t; \qquad (14)$$

$$\sqrt{2}\,\sigma_1^{t^{1/2}}+\sigma_2^{t^{1/2}}\geqslant 2\sigma_3^{t^{1/2}}\geqslant|\sqrt{2}\,\sigma_1^{t^{1/2}}-\sigma_2^{t^{1/2}}|;$$

$$\sqrt{2}\,\sigma_1^{t^{1/2}}+2\sigma_3^{t^{1/2}}\geqslant \sigma_2^{t^{1/2}}\geqslant|\sqrt{2}\,\sigma_1^{t^{1/2}}-2\sigma_3^{t^{1/2}}|;$$

$$\sigma_2^{t^{1/2}}+2\sigma_3^{t^{1/2}}\geqslant\sqrt{2}\,\sigma_1^{t^{1/2}}\geqslant|\sigma_2^{t^{1/2}}-2\sigma_3^{t^{1/2}}|.$$

Analyses of near threshold photoproduction of two π-mesons in deuterium without deuteron breakup, yields (here $V_{10}=0$):

$$\sigma(\gamma+d\to d+\pi^0+\pi^0)=\sigma(\gamma+d\to d+\pi^++\pi^-) \quad (15)$$

and for the total cross sections:

$$2\sigma^+(\gamma+d\to d+\pi^0+\pi^0) \qquad (16)$$
$$=\sigma^t(\gamma+d\to d+\pi^++\pi^-).$$

The inequalities relating the cross sections of the various processes and Eqs. (15)–(16) hold away from threshold if the π-mesons produced in reactions (1) and (10) are created with equal momenta [2] $K_1=K_2$.

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Translated by M.A. Melkanoff
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Paramagnetic Resonance and the Polarization of Nuclei in Thick Metal Foils

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ACCORDING to Overhauser[1], paramagnetic resonance induces polarization in nuclear matter. It is shown in Ref. 1 that a substantial nuclear polarization occurs in metals when the frequency

$$\Omega_0=\mu H_0/\hbar \gtrsim kT/\hbar, \quad \text{a} \quad \Omega_1=\mu H_1/\hbar\gtrsim 1/T_{sp}$$

(H_0 and H_1 are the constant and the high frequency magnetic field intensities, μ is the magnetic moment of the electron, T_{sp} is the spin relaxation time). The later estimate, however, is correct only when the electron moves in the time T_{sp} through a homogeneous electromagnetic field. At resonance ($\omega=\Omega_0$), when $\Omega_0 \gg kT/\hbar$, this takes place only in extremely thin metallic foils, of thickness of the order of the skin depth $\delta\sim 10^{-4}$ to 10^{-6} cm. So far[2], only such specimens have been polarized by Overhauser's method.

In the present article it is shown that it is possible by means of a high frequency magnetic field of high intensity

$$H_1\gtrsim(8\pi\delta_{\text{eff}}/c^2|Z|T_{sp})H_0$$

(Z is the surface impedance of the metal[3]), to polarize nuclei in thick foils at a considerably larger depth $\delta_{\text{eff}}\sim 10^{-2}$ to 1 cm to which electrons penetrate by diffusion in time T_{sp}.

In order to formulate a consistent theory it is necessary in this case to solve simultaneously Maxwell's equations

$$\text{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial B}{\partial t};\ \text{rot}\,\mathbf{H}_1=\frac{4\pi}{c}\mathbf{j};\ \mathbf{B}=\mathbf{H}_1+4\pi\mathbf{M},$$

and the kinetic equation for the electron distribution operator $\hat{f}$ ($\hat{f}$ operates only on spin)

$$\frac{\partial\hat{f}}{\partial t}+\frac{\partial\hat{f}}{\partial\mathbf{r}}\mathbf{v}+\frac{\partial\hat{f}}{\partial\mathbf{p}}\left\{e\mathbf{E}+\frac{e}{c}[\mathbf{vH}]\right\} \qquad (1)$$
$$+\frac{i}{\hbar}[\mu\mathbf{H}\hat{\sigma},\hat{f}]+\left(\frac{\partial\hat{f}}{\partial t}\right)_{col}+\left(\frac{\partial\hat{f}}{\partial t}\right)_{sp}=0.$$

$(\partial\hat{f}/\partial t)_{col}$ is here the collision integral for collisions without spin flip, $(\partial\hat{f}/\partial t)_{sp}$ is the integral for collisions with spin flip; $\hat{\sigma}$ is the spin operator; **v** and **p** are the velocity and momentum of the electron. Let us write these integrals in the form

$$(\partial\hat{f}/\partial t)_{sp}=(\hat{f}-\hat{f}_0)/T_{sp};\quad(\partial\hat{f}/\partial t)_{col}=(\hat{f}-\hat{f}^0)/t_0$$

(t_0 is the time between collisions), where

$$\hat{f}_0 = \begin{pmatrix} f_0(\varepsilon_0 + \mu H_0) & 0 \\ 0 & f_0(\varepsilon_0 - \mu H_0) \end{pmatrix};$$

$$f^{\hat{0}} = \int \hat{f} \frac{dS}{v} \Big/ \int \frac{dS}{v}; \; f_0(\zeta) = [e^{(\varepsilon - \zeta)/kT} + 1]^{-1}.$$

In order to determine $\hat{f}^0$, let us compute what such an equilibrium operator would become for the given $\hat{f}$ in the absence of processes which bring about spin flip (i.e., for $\Omega_1 = 0$, $T_{sp} = \infty$). Evidently then,

$$\int \hat{f}\, dp_x\, dp_y\, dp_z = \int \hat{f}^0\, dp_x\, dp_y\, dp_z,$$

$$\mathrm{Sp} \int \hat{f}\, dp_x\, dp_y\, dp_z = \mathrm{Sp} \int \hat{f}_0\, dp_x\, dp_y\, dp_z$$

(ϵ_0 is the Fermi energy level; dS is an element of the Fermi surface).

Boundary conditions for the function f at the metal surface have the form*

$$\hat{f}\,\big|_{\mathbf{vn}>0} = (1 - q)\hat{f}^0 + q f\,\big|_{-\mathbf{vn}} \tag{2}$$

($\mathbf{n}$ is the internal surface normal, q is the coefficient for electron reflection from the surface; apparently, almost always, $q \approx 0$).Expanding $\hat{f}$ in terms of the operators $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\hat{\sigma}$, it may be shown that Eq. (1) corresponds to the following system of equations:

$$\frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \mathbf{r}} \mathbf{v} + \frac{\partial f_1}{\partial \mathbf{p}} \left\{ \frac{e}{c} [\mathbf{vH}] + e\mathbf{E} \right\} \tag{3}$$

$$+ \frac{f_1 - f_0(\varepsilon - \varepsilon_0)}{t_0} = 0,$$

$$\frac{\partial \mathbf{w}}{\partial t} + \frac{\partial \mathbf{w}}{\partial \mathbf{r}} \mathbf{v} + \frac{e}{c} [\mathbf{vH}] \frac{\partial \mathbf{w}}{\partial \mathbf{p}} + \frac{\mathbf{w}}{t_0^*} + [\mathbf{w}\Omega] = \frac{\overline{\mathbf{w}}}{t_0} + [\mathbf{b}\Omega_1], \tag{4}$$

$$\mathbf{w}\,\big|_{\mathbf{vn}>0} = (1 - q)\overline{\mathbf{w}} + q\mathbf{w}\,\big|_{-\mathbf{vn}},$$

$$f_1\,\big|_{\mathbf{vn}>0} = (1 - q) f_0 + q f_1\,\big|_{-\mathbf{vn}} \quad \text{(see Ref. 7)},$$

$$\frac{1}{t_0^*} = \frac{1}{t_0} + \frac{1}{T_{\text{сп}}}, \; \mathbf{b} = \frac{\mathbf{H}_0}{H_0}, \; \overline{\mathbf{w}} = \int \mathbf{w} \frac{dS}{v} \Big/ \int \frac{dS}{v}, \tag{5}$$

where

$$\mathbf{j} = 2eh^{-3} \int f_1\, dp_x\, dp_y\, dp_z, \; \mathbf{M} = \chi H_0 (\mathbf{b} - \overline{\mathbf{w}}).$$

It is easily seen that if $T_{sp} = \infty$, $\omega = \Omega_0$, Eq. (4) has a non-trivial solution for $\Omega_1 = 0$, depending only upon ϵ. But $T_{sp} \gg t_0$; we therefore find ourselves near the eigenvalue of Eq. (4). Its solution, therefore, turns out to be near the corresponding eigenfunction (independent of $\mathbf{r}$) and slowly varying as a function of the depth $\xi = \mathbf{nr}$.

At resonance (for $\omega = \Omega_0$), solving the equation, we obtain:

$$\overline{w}_z(\xi) = \frac{|\alpha|^2}{1 + |\alpha|^2} \exp\{-\xi/\delta_{\text{eff}}\}, \tag{6}$$

$$\overline{w}_x + i\overline{w}_y = -\frac{\alpha e^{i\omega t}}{1 + |\alpha|^2} \exp\{-\xi/\delta_{\text{eff}}\},$$

$$(t_0/T_{sp})^{1/2} (a/\delta)^{4/3} \ll 1,$$

where

$$\alpha = \frac{c(E_x + iE_y)}{2H_0 a} \sqrt{t_0 T_{sp}}, \; \delta_{\text{eff}} = a\sqrt{T_{sp}/t_0} \tag{7}$$

(the z-axis coincides with the direction of the constant field H_0, the y-axis lies in the surface of the metal), $\mathbf{E}$ is the amplitude of the electric field at the surface of the metal, and the quantity a is of the order of magnitude of the orbital radius of the electron $r_0 \sim mvc/eH_0$ in the case of a strong ($r_0 \ll vt_0$) magnetic field H_0 parallel to the surface, and of the order of vt_0 in all other cases.

It is easy to obtain from formulas (6) the nuclear polarization P:

$$P = I^{-1} \{ (I + 1/2) \operatorname{cth} (I + 1/2) s - 1/2 \operatorname{cth} (s/2) \},$$

$$s = \frac{|\alpha|^2}{1 + |\alpha|^2} \frac{\mu H_0}{kT} e^{-\xi/\delta_{\text{eff}}}$$

(I is the magnetic moment of the nucleus).

Let us note that a slow damping of the magnetic moment $\mathbf{M}$ [see Eqs. (6) and (7)] brings about, according to Maxwell's equations, the appearance of a slight, though equally slow, damping of $\mathbf{E}$ and $\mathbf{H}_1$**.Thus, in paramagnetic resonance one should observe the resonance passage of an electromagnetic wave through the film, whereby the transmitted wave becomes circularly polarized.

The transmission coefficient $K = |E_{tr}|^2/|E_{inc}|^2$ for a film of thickness d ($\delta \ll d \ll \delta_{\text{eff}}$), is of the order of magnitude

$$K = \left(\frac{cT_{sp}\, \chi\, |cz|^2}{2\pi d} \right)^2 \left[1 + \left| \frac{c^2 z T_{sp} H_{\text{inc.}}}{2\pi d H_0} \right|^2 \right]^{-2} (\mathbf{n} \parallel z),$$

i.e., it can be many times larger than the transmission coefficient away from resonance. It should be noted that such a selective transparency of a film occurs at any temperature.

The derivations of the results reported here will be the subject of a separate article.

The author is indebted to L. N. Rosentsveig for discussion of the results of this investigation.

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* It is assumed that the electron spin is not flipped upon reflection. Let us note that Eq. (1) was obtained in Ref. 4; the operator $(\partial f/\partial t)_{sp}$, however, was not included here. Boundary conditions (13) of that reference are apparently not realizable in practice.

** The slow damping of **M** is linked to the fact that when $\delta \ll \delta_{eff}$ there appears an "anomalous skin-effect" for the magnetic moment: an integral relation appears between M and H_1. This does not take place out of resonance as $M \sim 10^{-6} H_1$ (while at resonance $M \sim 10^{-2} H_1$).

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"Repulsion" of Nuclear Levels

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1. DATA on the position and parameters of nuclear levels obtained by methods of neutron spectroscopy at excitation energies of the order of the neutron binding energy[1,2] allow the investigation of the empirical regularities of level behavior with the aim of checking the predictions of existing nuclear theories and their improvement. Interesting work has been done, for example, on the systematics of neutron widths[3] and on the systematics of radiation widths[4] *. The problem of regularities in the distribution of nuclear levels and in the fluctuations of adjacent level spacing has not been discussed in the literature.

For a purely random distribution of the distance between levels ϵ about its average value D, the distribution function must be of the form:

$$W(\varepsilon)\, d\varepsilon = \exp\{-\varepsilon / D\}\, d\varepsilon / D. \qquad (1)$$

It would be more to the point to examine the data for levels in the same spin state, i.e., for target nuclei of spin 0 (even-even nuclei). However, such nuclei have few levels in the range where the resolution of contemporary methods of nuclear spectroscopy suffices. Thus it is impossible to exclude the data obtained from nuclei with odd atomic weight, and consequently with two sets of nuclear levels, corresponding to $i - 1/2$ and $i + 1/2$ (where i is the spin of the target nucleus). It should be kept in mind that the presence of two sets of levels corresponding to various spins of the intermediate nucleus makes the correlation of various levels positions less obvious.

If the distribution function (1) holds for each set of levels, then the resulting distribution function will have the same form with $D = d_1 d_2 / (d_1 + d_2)$, (where $d_{1,2}$ are the distances between levels in each set).

We made use of experimental data on the level distribution for: In^{113}, In^{115}, Cs^{133}, Tb^{159}, Ho^{165}, Tm^{169}, Hf^{177}, Hf^{179}, Ta^{181} [1], U^{235} **, U^{238} [1].

In order to eliminate mistakes in the determination of ϵ due to ignoring levels because of insufficient experimental resolution, a curve was constructed for each element of the increase of the number of discovered levels with increase of neutron energy. Levels within a suitable energy limit were used, so that this increase was approximately linear.

To increase the statistical certainty of the experimental distribution of levels for each isotope, the quantity $x_i = \epsilon_i / D$ was calculated, and the distribution of levels as a function of x_i for all the enumerated nuclei was plotted (Fig. 1). The total number of cases $N = 134$. The curve is the distribution (1) normalized to the area of the histogram.

The level distributions for U ^{238} (an even-even nucleus with eleven known levels) and for U ^{235} (for which D is comparable to the entire level width) are shown separately.

Comparison of the curve and the histogram allows the qualitative confirmation of the relatively small number of closely spaced levels, which may be interpreted as the result of a "repulsion" of levels.

To eliminate the possibility that the relatively small number of closely spaced levels is due to insufficient resolving power of the experiments, we subjected the data to a stricter selection. The following were excluded from consideration: a) csses in which there might be some doubt as to which isotope a given level belongs, b) levels for which the full width of the resolution function ΔE does not satisfy the condition $\Delta E < 0.2D$. The total number of available cases was thus reduced from 134 to 63, of which 19 belong to U^{235} (assuming that the resolution of the Brookhaven National Laboratory experiments were 0.07 μ sec/meter), 7 to U^{238}, and 37 to Cs^{133}, Ho^{165}, Tm^{169}, Hf^{177}, Hf^{179}, Ta^{181}.

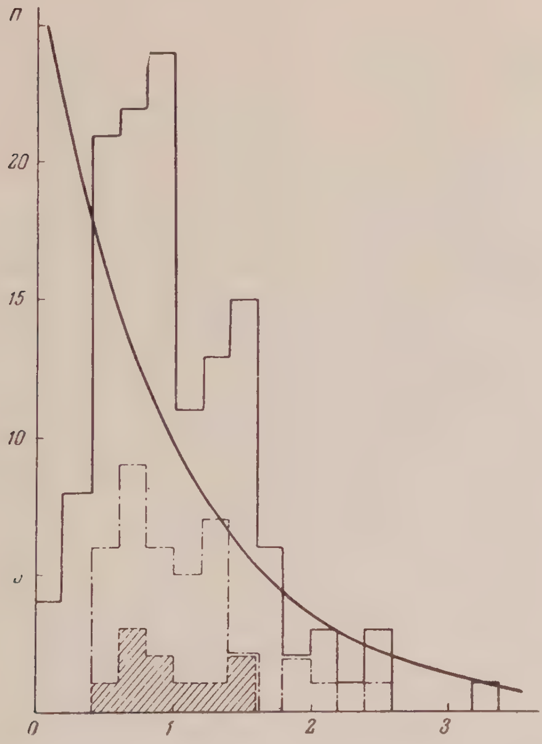

FIG. 1. Distribution of levels according to $x = \epsilon/D$. The shaded histogram $-U^{238}$; — . — corresponds to U^{235}; — to the sum. The curve corresponds to distribution (1) normalized to the area of the sum histogram.

Figure 2 shows the respective histograms. A general view of the histograms is still characterized by a relatively small number of cases with low x.

In order to determine whether the discrepancy between the experimental results and the distribution (1) is not coincidental, χ^2 was calculated for 63 cases, the distribution being divided into 9 groups. The magnitude of χ^2 was 27.5, and the corresponding probability of a coincidental discrepancy $P(\chi) < 0.001$.

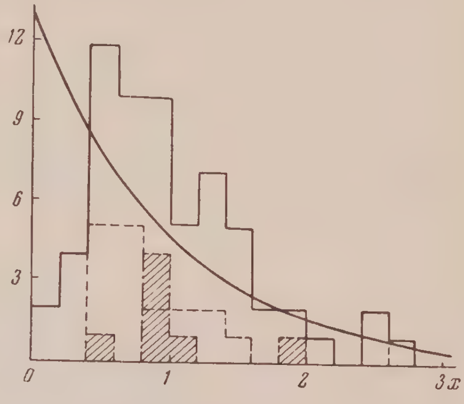

FIG. 2. The same as Figure 1, after additional selection.

The occurrence of level "repulsion" is seen also in the magnitude of the mean square fluctuation of the level spacing $\overline{\Delta x^2} = \overline{x^2} - \bar{x}^2$, $(\overline{\Delta\epsilon^2} = D^2\,\overline{\Delta x^2})$. While $\overline{\Delta x^2} = 1$ for distribution (1), in our case $\overline{\Delta x^2} = 0.31$ (for all 63 cases).

The interaction of nuclear levels which we have examined are similar in their physical nature to the phenomenon of electronic term crossings in the spectra of diatomic molecules. A quantum mechanical view of the molecular term behavior[5] shows that their crossing, i.e., the superposition of two terms, cannot take place in the general case.

As can be seen from the above analysis, the interaction of nucleons in the nucleus leads to a distribution function of the nuclear levels which is closer to equidistant than would be predicted from statistical theory.[6]

2. An overwhelming part of the experimental data on the distance between levels is for two systems of levels of the same parity and various spin. In the interpretation of such data, two limiting assumptions can be made: a) the interactions between levels of the two systems are the same as between levels of a single system; b) levels of the two systems do not interact and are randomly distributed (we regard the second assumption as more probable).

In the case that assumption b) is valid, if the level density of one system is much greater than that of the other system, then the experimental distribution of the level distances [as also for assumption a)] will be practically that of one of the level systems. Consequently, if enough statistical material is obtained, the distribution function $\omega(\epsilon)$ for the particular system can be obtained.

The second situation for b), when the level densities of the two systems do not differ appreciably, can be analyzed in the following manner. Let ω_1 (ϵ) and ω_2 (ϵ) be the distribution functions for the distance between levels in each of the two systems. $[\omega_i = d_i^{-1} f(\epsilon/d_i)]$. The probability of finding a level of the second system at a distance ϵ from a level of the first system is given by the expression

$$\varphi_i(\varepsilon) = \frac{1}{d_i}\int_{\varepsilon}^{\infty} w_i(\eta)\, d\eta.$$

It is easy to show that the distribution function for the distance between levels for the entire system (levels are not distinguished) will be

$$W(\varepsilon) = D d^2\Psi(\varepsilon)/d\varepsilon^2, \tag{2}$$

where

$$\Psi(\varepsilon) = \psi_1(\varepsilon)\,\psi_2(\varepsilon),$$

$$\psi_i(\varepsilon) = \int_{\varepsilon}^{\infty} \varphi_i(\eta)\, d\eta = \frac{1}{d_i}\int_{\varepsilon}^{\infty} (\eta - \varepsilon)\, w_i(\eta)\, d\eta, \tag{3}$$

D is the average distance between levels of the entire system $[\psi_i\ (0) = 1]$.

If we assume that the repulsion of levels for each system must make $\omega_{1,2}$ (0) go to zero, it follows from Eq. (2):

$$W(0) = \frac{2D}{d_1 d_2} = \frac{1}{D}\,\frac{2\alpha}{(1+\alpha)^2}, \tag{4}$$

where $\alpha = d_1/d_2$ the ratio of the level distances of the two systems.

The presently available statistical material is not sufficient to allow detailed analysis along the above lines.

In conclusion we thank S. T. Beliaiev and V. M, Galitskii for fruitful discussions about two systems of related levels.

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* The work of Levin and Hughes[4] was sent to the USSR before its publication, for which we thank the authors.

** Unpublished data from Brookhaven National Laboratory. We thank V. V. Vladimirskii for communicating these results, obtained from Prof. D.H. Hughes.

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31

The Polarization of Neutrons from the $D(\gamma, n)$ Reaction

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THE derivation of equations which give the cross section for photodisintegration of the deuteron for $E_\gamma \lesssim$ 10 mev can be found in any textbook of nuclear physics (for example, Ref. 1), but there is no mention in the literature that the products of this process are polarized. As is shown below, this effect is caused by an interference of electric and magnetic transitions.

The polarization of the neutrons $\zeta = \langle \sigma_n \rangle$ from the $D(\gamma, n)$ reaction is

$$\zeta = \mathrm{Sp}\,(\sigma_n Q_\lambda Q^{+}_{\lambda'}) \times (\delta_{\lambda\lambda'} + \xi\Omega_{\lambda\lambda'})/\mathrm{Sp}\,(Q_\lambda Q^{+}_{\lambda'})\,(\delta_{\lambda\lambda'} + \xi\Omega_{\lambda\lambda'}), \tag{1}$$

where Ω has the components

$$\Omega_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad \Omega_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad \Omega_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2}$$

ξ_x, ξ_y, ξ_z are parameters characterizing the polarization of the incident bundle of photons (Ref. 2), and Q_λ is the matrix element for the photodisintegration of the deuteron by a γ-quantum of momentum $\mathbf{k} = \omega\varkappa$ $(|\varkappa| = 1)$* and polarization vector $e_\lambda (\lambda = 1,2)$

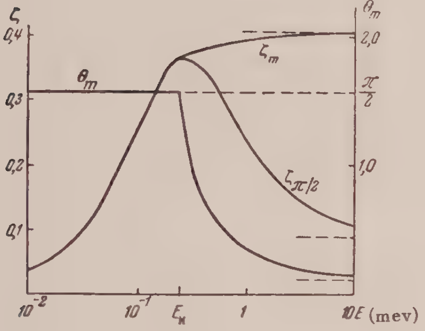
In the approximation of central $n-p$ forces with zero interaction radius and using the usual notation

$$Q_\lambda \sim \delta_{S1}\delta_{mm_0}\frac{(\mathbf{p}\cdot\mathbf{e}_\lambda)}{\alpha_1^2 + p^2} + \delta_{S0}\delta_{m0}\, i\,(\mu_p - \mu_n)\,(\chi_0, ([\varkappa\cdot\mathbf{e}_\lambda], \sigma_p - \sigma_n)\,\chi_1^{m_0})\,\frac{\alpha_1 - \alpha_0}{4M(\alpha_0 + ip)}. \tag{3}$$

We have taken into account only dipole transitions and have dropped terms not important to Eq. (1).

Substitution of expression (3) into (1) gives for the unpolarized bundle of γ-quanta

$$\zeta = \frac{{}^1/_3\,(\mu_p-\mu_n)\sin\theta\sqrt{M}\,(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_0})\,E/(\varepsilon_0+E)(\varepsilon_1+E)}{ME\,(\varepsilon_1+E)^{-2}\sin^2\theta+{}^1/_6\,(\mu_p-\mu_n)^2(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_0})^2/(\varepsilon_0+E)}\,\mathbf{n}, \qquad (4)$$



θ_m is the angular departure corresponding to maximum polarization of ζ_m of photoneutrons, $\zeta_{\pi/2}$ is the polarization of neutrons departing at an angle of $\pi/2$ (for $E < E_k$, $\theta_m = \pi/2$ and $\zeta_m = \zeta_{\pi/2}$).

$[\varkappa \cdot \mathbf{p}]$.

where $\mathbf{n}$ is the unit vector in the direction $\varkappa \times \mathbf{p}$. In the vicinity of the threshold the maximum value of $|\zeta|$ is obtained for the angle $\theta = \pi/2$. However if the energy of the photoneutron E is higher than $E_k = 0.24$ mev**, then the maximum value of polarization is

$$\zeta_m = (E/6)^{1/6}(\varepsilon_0+E)^{-1/2} \approx 6^{-1/2}(1-\varepsilon_0/2E) \qquad (5)$$

corresponding to the emission angle $\theta = \theta_m$, where

$$\sin\theta_m \qquad (6)$$

$$= (\mu_p-\mu_n)(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_0})(\varepsilon_1+E)/\sqrt{6ME\,(\varepsilon_0+E)}$$

$$\approx 0{,}11\,(1+\varepsilon_1/E).$$

In the case of a polarized photon bundle, the vector $\mathbf{n}$ in Eq. (4) should be replaced by the sum

$$\mathbf{n}\,(1+\xi_x\cos 2\varphi+\xi_y\sin 2\varphi)$$

$$+\,[\mathbf{n}\varkappa]\,(\xi_x\sin 2\varphi-\xi_y\cos 2\varphi+\xi_z\sqrt{\varepsilon_0/E}),$$

and the first term in the denominator of Eq. (4) should be multiplied by $1+\xi_x\cos 2\varphi+\xi_y\sin 2\varphi$, where φ is the azimuthal angle of neutron emission.

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*We assume $\hbar = c = 1$.

**E_k is the root of the equation

$$6ME\,(\varepsilon_0+E) = (\mu_p-\mu_n)^2(\sqrt{\varepsilon_1}+\sqrt{\varepsilon_0})^2(\varepsilon_1+E)^2;$$
$$\varepsilon_1 = 2{,}23 \text{ mev}, \quad \varepsilon_0 = 0.064 \text{ mev};$$

for numerical evaluations we take $\epsilon_1 = 2.23$ mev, $\epsilon_0 = 0.064$ mev.

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New Type of Disintegration of a Heavy Meson?

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IN the summer of 1955, I. I. Gurevich and coworkers exposed an emulsion stack in the stratosphere. The stack consisted of 45 layers of type P emulsion. Thickness of an undeveloped layer was 400 μ, and diameter was 100 mm. The exposure was made at a height of 25 to 27 km, and the stack was at this height for two hours. Subsequently this emulsion stack was turned over to us by I. I. Gurevich. The result of microscopic scanning of the developed emulsions has been three cases, which are described below.

First Case (found by scanner N. G. Petruzashvilyi). The incident particle of unknown mass, having a path in emulsion of 2000 μ, stops and decays into a π^+-meson which leaves a track of length 365 μ. In its turn, the π^+-meson decays into a μ^+-meson with track length 630 μ, and finally decays into a positron. The whole chain of decays lies in the plane of a single emulsion layer.

Second Case (found by scanner K. A. Abashidze). The incident particle of unknown mass, emitted from a star which has 4 black and 3 relativistic tracks, travels for 5600 μ and decays into a π^--meson with range 353 μ, forms a σ-star consisting of three protons. The decay of the unknown mass particle and the σ-star are in a single emulsion layer.

Third Case (found by scanner L. N. Gabunia). The incident particle of unknown mass, having a range in emulsion of 6500 μ, stops and decays into a π^+-meson, which has a range 354 μ. The π^+-meson in its turn, decays into a μ^+-meson, which after 570 μ decays by the emission of a positron. The entire chain of decays lies in a single emulsion layer.

If we had only a single case to deal with, it would undoubtedly be interpreted as the decay of a τ'-meson, according to the scheme

$$\tau'^{\pm} \to \pi^{\pm} + 2\pi^0.$$

However, as can be seen from the description of these cases, the common characteristic for all is the occurrence of a π-meson track of length 357 μ ± 2%. Since these π-mesons are monochromatic, the decay of the ± particle of unknown mass is with high probability a two particle process.

A two particle decay of an incident meson into a π-meson of 3.4 mev energy (corresponding to 357 μ), is so far unknown.

The gradient of emulsion grains along the tracks of the unknown initial particles and also the nature of their multiple scattering does not allow us to differentiate between the possibilities of a fork caused by the decay of a neutral meson, or a two prong star, or a sudden change of direction of the initial particle in a single scattering event.

Since the exact measurement of initial particle mass by one of the indirect methods was made difficult by the inconvenient placement of the tracks relative to the emulsion, we are at present limited to an examination of various possible decay modes using particles of known mass.

Variation I. The decay scheme of the unknown particle is

$$?^{\pm} \to \pi^{\pm} + \pi^0 + Q.$$

Then its mass is

$$m_?^{\pm} = 560\ m_e,\ Q = 6.8\ \text{mev}.$$

Variation II. The decay scheme is

$$?^{\pm} \to \pi^{\pm} + \theta^0 + Q.$$

Then

$$m_?^{\pm} = 1260\ m_e,\quad Q = 4{,}4\ \text{mev}.$$

Variation III. The decay scheme is:

$$?^{\pm} \to \pi^{\pm} + \nu + Q.$$

Then

$$m_?^{\pm} = 350\ m_e,\quad Q = 33.4\ \text{mev}.$$

Variation IV. A K±-meson of mass 970 m_e decays with the scheme

$$K^{\pm} \to \pi^{\pm} + ?^0 + Q.$$

Then $$m_?^0 = 680\ m_e,\quad Q = 4.8\ \text{mev}.$$

It is interesting to note that three of the suggested variations give new masses for the initial incident particle, while the fourth gives a new value of mass for the neutral secondary meson. Attention should be directed to the fact that in one of the cases a negative initial particle stopped in the emulsion and was not captured by a nucleus, but decayed into a π^--meson which in turn formed a σ-star.

Indirect measurement of the incident particle mass continues.

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Connection between α-Decay and Nuclear Deformation

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In this communication a connection between the deformation of the nuclear surface[1] and the relative intensities of α-groups in complex α-spectra of radioactive nuclei will be established; the results calculated apply to the α-spectrum of RdAc.[2] Among those factors which influence the intensity of α-groups should be listed the exponential factor in the well-known Gamow-Bethe formula giving the coefficient for the passage of the α-particle through the potential barrier (penetration factor) and the factor in front of the exponential, expressing the probability of "barrier-less" emission of the α-particle by the radioactive nucleus. We note that the application of the exact Gamow-Bethe formula[3] with a fixed value of the radius R of the nuclear potential well does not give essentially better results than the preceding work of the author[4] obtained with approximate formulas. The role of the factor in front of the exponential is not essential in the questions of interest here with either the one-body model[2] or in the many-body model.[3]

It is of interest here to consider the influence of nuclear deformation on the intensity of α-groups in connection with the work of Ref. 5. In that work a modified formula was obtained for the probability of penetrating the barrier:

$$w = w_0 \exp[-S(E, R)]; \qquad (1)$$

$$S(E, R) = (8Ze^2/\hbar v)\left[\arccos\sqrt{x} - \sqrt{x(1-x)} - {}^4/_5\sqrt{x(1-x)}\,\beta_{\max}\right];\ x = E/V.$$

The difference from the usual formula consists in the introduction of the term with the relative deformation β ($\beta_{\max} \approx \Delta R/R$ at the place of the largest compression or extension of the nuclear surface). The remaining notation is: Ze is the electric charge of the daughter nucleus, v and E are the velocity and energy of the α-particle, V is the maximum height of the potential barrier. It is sufficient to allow the perturbation of the daughter nucleus by the emitted α-particle to produce a change in β relative to the ground level of only 10% of the value of β in order to obtain a change in the intensity by a factor of two or three, and this is a very attractive feature. In the single-particle and many-particle models of the nucleus the orders of magnitude of β are completely different. In the former, the α-particle moves in a constant mean field of the remaining nucleons of the nucleus, which produce a potential well with very sharp sides; the changes in the width of the well for comparitively small changes of the α-particle energy ΔE cannot be at all important. In the collective model of the nucleus changes in the deformation β of the required order are entirely conceivable, because with a sufficiently strong coupling between the nucleons and the nuclear surface, the parameters characterizing the potential well will adiabatically follow the changes in state of the nucleus and alpha-particle. In the transition from the ground state to the excited one, a change in β ($\Delta\beta$) will be brought about by the centrifugal force which arises as a result of the rotation of the nucleus, the mechanical moment of which is determined by the resultant moment of all excited rotators l (see Ref. 4). Therefore it is permissable to use the dependence for $\Delta\beta$ obtained in Ref. 6.

$$\Delta\beta = (\eta K/C)\, l\,(l+1), \qquad (2)$$

where C is the surface coefficient equal to $M_2\omega_2^2 R_c^2$ (M_2 is the effective mass of the vibration of the second vibrational harmonic, ω_2 is the frequency of this vibration and R_c is the nuclear radius) and K is the coefficient of the first-order (in β) coupling of the nucleon with the nuclear surface

$$K = k\sqrt{5/4\pi}\,[3l^2 - j(j+1)]/4j(j+1), \qquad (3)$$

where k is a constant of the coupling with the surface, l is the quantized moment of the α-particle and j is the change in the moment of the nucleon which is outside the shell and which participates in the balance of energy and momentum of the system; η is a numerical coefficient equal —in order of magnitude— to several hundred units.

The size of K in Eq. (2) cannot be immediately identified with the coupling constant k. In fact, in the derivation of Eq. (2) it was necessary to equate the generalized force $C\beta$, which comes from the change of the nuclear surface as a result of deformation, to the generalized force of the centrifugal potential; in view of this the factor $d\,\Delta R/d\beta$ enters into the right-hand side of Eq. (2). This can be calculated, for example, by writing the expression for the quadrupole moment on one hand, through[7] ΔR and on the other hand, through[8] β , after which it is easy to convince oneself that the above argument is correct. These calculations show that the factor taking into account the influence of nuclear deformation in Eq. (1) does not change the statistical weights of the states with $l=0$ (main group), but considerably increases the statistical weights of the states with $l > 2$ and sharply decreases them in the states with $i < 2$ (under the condition $k > 0$). Choosing , in accordance with Ref. 9, $C=65$ mev, $k = 25$ mev and $R = 10^{-12}$ cm, we find that the calculation of nuclear deformations together with the decay scheme of Ref. 4 significantly improves the agreement with experiment,[10] which can be considered as an indirect confirmation of the collective (generalized) model of the nucleus.

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On the Probabilities of Σ-Particle Decay

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RECENTLY Gatto[1] indicated the interest in applying arguments developed first by Fermi[2] for the photoproduction of π-mesons to the problem of the decay of "strange" particles. As is well known, Fermi showed, on the basis of the requirements of unitarity and symmetry of the S-matrix, that the phases of strongly interacting particles which are produced in weak processes are defined by their mutual scattering. Gatto considered the decay of the Λ-particle from this point of view and came to the conclusion that the restrictions imposed by the unitarity and symmetry of the S-matrix on the ratio of probabilities of the two possible modes of decay of the Λ-hyperon ($\Lambda \to p + \pi^-$ and $\Lambda \to n + \pi^0$) are very weak. This conclusion, as will be evident from the following, depends to a large extent on the fact that the scattering of π-mesons by nucleons at energies ~40 mev is still small and the corresponding phases are small.

We consider in analogous fashion the decay of the Σ-hyperon. In this decay, ~ 115 mev is given off, corresponding to the scattering of π-mesons of ~ 140 mev in the laboratory system by nucleons. It is well known that the scattering of π-mesons by nucleons is already considerable (according to Orear[3] the phases $\alpha_{33} \sim 40°$, $\alpha_3 \sim -10°$, $\alpha_1 \sim 15°$, $\alpha_{31} = \alpha_{13} = \alpha_{11} = 0$; the d-phases do not exceed 5°).

The π-meson-nucleon system formed as a result of Σ-particle decay, has the following values J of total and L of orbital angular momentum, depending on the spin S and parity P of the hyperon:

Spin and parity of the Σ-hyperon		State of the $\pi + N$ system		
S	P	J	L	Phases
1/2	+	1/2	1	α_{31}, α_{11}
1/2	—	1/2	0	α_3, α_1
3/2	+	3/2	1	α_{33}, α_{13}
3/2	—	3/2	2	

(P designates the parity of the Σ-hyperon relative to the nucleon). Thus, L and J of the π-meson-nucleon system are determined unambiguously by the spin and parity of the Σ-hyperon.

We consider two possible decay modes for the Σ-hyperon:

$$\Sigma^+ \to p + \pi^0 \text{ and } \Sigma^+ \to n + \pi^+.$$

If weak interactions are excluded, then the S-matrix has only diagonal elements, transforming Σ^+ into Σ^+ and the π-meson-nucleon system with a given T into a state with the same T. These diagonal elements are equal to 1, $e^{i2\alpha_3}$ and $e^{i2\alpha_1}$ (the second index of the phases α_3 and α_1 is omitted).

If weak interactions are now included, nondiagonal elements arise which transform the Σ-particle into states of the π-meson-nucleon system with $T = 3/2$ and $1/2$. It follows from the unitarity and symmetry of the S-matrix [2] that these diagonal elements have the form $i\rho_3\, e^{i\alpha_3}$ and $i\rho_1\, e^{i\alpha_1}$, where ρ_3 and ρ_1 are real. The transition amplitude for the Σ^+-hyperon going into the states $\pi^+ n$ and $\pi^0 p$ can be expressed in the following way in terms of the isotopic amplitudes

$$a_+ = \sqrt{\frac{1}{3}}\, i\rho_3 e^{i\alpha_3} + \sqrt{\frac{2}{3}}\, i\rho_1 e^{i\alpha_1},$$
$$a_0 = \sqrt{\frac{2}{3}}\, i\rho_3 e^{i\alpha_3} - \sqrt{\frac{1}{3}}\, i\rho_1 e^{i\alpha_1}.$$

The ratio of corresponding decay probabilities is

$$X = \frac{w_0}{w_+} = \frac{2 + 2z^2 - 4z\cos(\alpha_1 - \alpha_3)}{1 + 4z^2 + 4z\cos(\alpha_1 - \alpha_3)},$$

where

$$z = \rho_1 / \rho_3 \sqrt{2}.$$

The phase differences for various spins and parities of the Σ-hyperon are: $(1/2+)\ \alpha_{11} - \alpha_{31} \approx 0$; $(1/2-)\ \alpha_1 - \alpha_3 \approx 25°$; $(3/2+)\ \alpha_{13} - \alpha_{33} \approx -40°$.

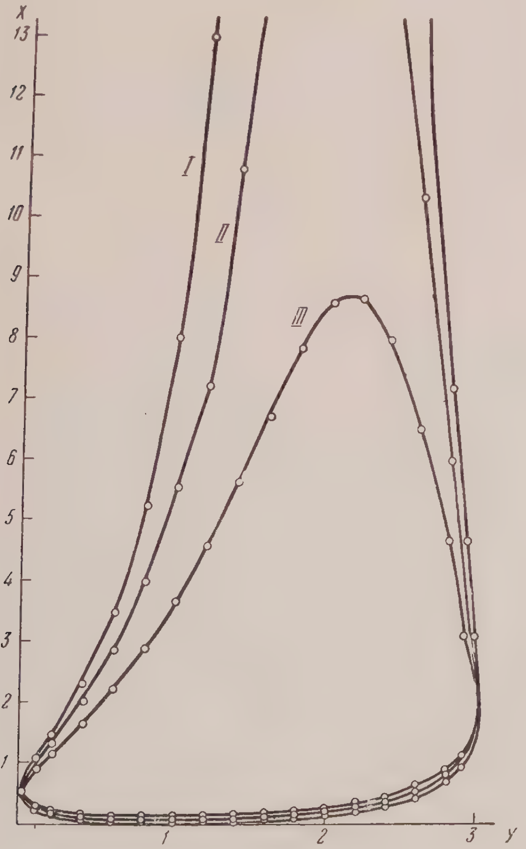

Figure 1

For all other values of the spins and parities the differences of the corresponding phases (and the phases themselves) are close to zero. The limits of X depend on the phase differences in the following way

Difference in phase	0	~25°	~40°
Upper limit of X	∞	~20	~10
Lower limit of X	0	~1/20	~1/10

The restrictions obtained on the magnitude of X are very weak. None the less, if it turns out that $X > 10$ or $X > 1/10$, this will mean that the Σ-particle definitely does not belong to the class $3/2+$, and if $X > 20$ or $X < 1/20$, then the possibility $1/2-$ is unambiguously excluded.

The ratio of probabilities of different Σ^+-hyperon decays can be related to the decay probability of the Σ^--hyperon if it is assumed, as Gatto[1] has done, that the interaction leading to the decay is a tensor of rank 1/2 in isotopic space. It is easy to show that in this case the decay amplitude for $\Sigma^- \rightarrow n + \pi^-$ is equal to

$$a_- = i\sqrt{3}\,\rho_3 e^{i\alpha_3};$$

where ρ_3 is the same quantity as in the expressions for a_+ and a_0.

The ratio of the decay probabilities for the Σ^--particle and Σ^+-particle is, in this case, equal to

$$Y = \frac{w_-}{w_+ + w_0} = \frac{3}{1 + 2z^2}.$$

We find in this way that for every possible value of the spin and parity of the Σ-particle, the points (X, Y) can lie only on a well-defined curve in the XY plane. Gatto, who did not use the unitarity and symmetry of the S-matrix in considering the Σ-decay, obtained in this case a considerably less restrictive result; he found that the points (X, Y) can cover a certain allowed region — which is one and the same for all values of the spin and parity of the Σ-particle. If the phase difference is close to zero (that is, in all cases except those in which the spin and parity of the Σ-particle are equal to $1/2-$ or $3/2+$) the curve $Y(X)$ will be close to the curve bounding the allowed region of Gatto's work[1] (see Fig. 1). The curves for the phase differences of 25 and 40° are shown also on this Figure (curves II and III, respectively).

If the point (X, Y) characterizing the values of the decay probabilities as found by experiment lies on one of these two curves, this will signify (if we neglect the improbable possibility of a chance coincidence) the correctness of Gatto's hypothesis and will make it possible to draw definite conclusions about the spin and parity of the Σ-hyperon.

I wish to express my sincere gratitude to I. Iu. Kobzarev for interesting and useful discussions and to Prof. I. Ia. Pomeranchuk for the interest shown by him in this work.

Note added in proof. After this article was submitted, the author learned of the work in Refs. 4–6 in which the essentials of the above results are contained. In addition to this, preliminary experimental results were announced at the Sixth Rochester Conference which indicated that the magnitude of X was near to 1, and that of Y near to 0.1 –0.2. As is evident from the Figure, these data agree with the assumption that $\Delta T = 1/2$.

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Relation between the Parameters of Longitudinal and Transverse Optical Vibrations of Ions in Crystals

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LET us consider vibrations with wavelengths much larger than the lattice constant, but smaller than $c/\nu \sim 10^{-3}$ cm. (ν is the characteristic frequency for infrared dispersion in the crystal). The latter assumption enables us to treat the electromagnetic field created by the vibrating ions as electrostatic, i.e., to neglect the retardation effects as well as the effect of the magnetic field.[1,2] For this range of wavelengths one can also neglect the dispersion of the vibration eigenfrequencies.

In isotropically polarizable ionic (cubic) crystals, the polarization vibrations separate into transverse and longitudinal vibrations. We so normalize the normal coordinates that, for any transverse or longitudinal vibration, the Hamiltonian has the form $\frac{1}{2}(\dot{q}^2 + \Omega_j^2 q^2)$, where Ω_j is the eigenfrequency of the corresponding transverse vibration ($\Omega_{\perp j}$) or longitudinal vibration ($\Omega_{\parallel j}$); j is the branch number of the dispersion. Each normal vibration gives rise, in the crystal, to an inertial polarization, i.e., to a polarization due to the displacement of the ions and to the electron polarization produced by the displacement of the ions in the absence of an external field. The inertial polarization dipole moment density $\mathbf{p}(\mathbf{r}, t)$ varies sinusoidally in space, and its amplitude $\mathbf{p}_0(t)$ is proportional to q. Let $p_0 = \alpha_{\perp j}\, q$ for the transverse vibration and $p_0 = \alpha_{\parallel j}\, q$ for the longitudinal vibration. The relation between the parameters Ω_j, α_j for transverse and longitudinal vibrations is derived below.

Let us consider the forced vibrations of ions produced by an external electric field $\mathfrak{E}(\mathbf{r}, t)$. This field is chosen as a plane sinusoidal standing wave, vibrating harmonically in time with a frequency ω. Assuming that the interaction energy per unit volume of the crystal is equal to $-\mathbf{p}\mathfrak{E}$, we get for the dipole moment desnity due to forced vibrations:

$$\mathbf{p} = \sum_{j=1}^{s} \mathfrak{E}\alpha_j^2 / (\Omega_j^2 - \omega^2), \tag{1}$$

where s is the number of ions in the elementary cell of the crystal, minus one. The total polarization dipole moment due to the external field is $\mathbf{P} = \mathbf{p} + \mathbf{p}_e$ where $\mathbf{p}_e$ is the additional non-inertial polarization dipole moment due to the direct effect of the external field on the electron shells, the positions of the ions being held fixed. Let us consider two cases:

1) The external field is transverse and div $\mathbf{P} = 0$, i.e., the fictitious charges of dielectric polarization and their corresponding fields do not arise. In this case the external field $\mathfrak{E}$ coincides with the field $\mathbf{E}$ of macroscopical electrodynamics, and $\mathbf{p}_e = \mathbf{E}(n^2 - 1)/4\pi$, where n is the index of refraction for light in the crystal, in the plateau region of the dispersion curve— between the region of electronic absorption and the region where the absorption of infrared light by the vibration of the ions takes place. We get

$$\mathbf{P}=\mathrm{p}+\mathrm{p}_e=\left(\sum_{j=1}^{s}\frac{\alpha^2_{\perp j}}{\Omega^2_{\perp j}-\omega^2}+\frac{n^2-1}{4\pi}\right)\mathrm{E}. \qquad (2)$$

Identifying this formula with the formula of phenomenological electrodynamics: $\mathrm{P}=[\epsilon-1]\ \mathrm{E}/4\pi$ we get:

$$\varepsilon(\omega)/n^2=1+\sum_{j=1}^{s}a_{\perp j}/(\Omega^2_{\perp j}-\omega^2), \qquad (3)$$

$$a_{\perp j}=4\pi\alpha^2_{\perp j}/n^2.$$

In particular, if E is the field of the light wave (3) is the usual dispersion formula in the infrared region.

2)The external field is longitudinal. In this case the curls of the vectors p and p_e vanish, but their divergences do not. The field of the fictitious charge of dielectric polarization is $-4\pi\mathrm{P}$. It follows that the total macroscopic field is equal to $\mathrm{E}=\mathfrak{E}-4\pi\mathrm{P}$, i.e., the external field $\mathfrak{E}$ coincides with the displacement of macroscopical electrodynamics. Then $\mathrm{p}_e=\mathrm{D}(n^2-1)/4\pi n^2$ and

$$\mathbf{P}=\left(\sum_{j=1}^{s}\frac{\alpha^2_{\parallel j}}{\Omega^2_{\parallel j}-\omega^2}+\frac{n^2-1}{4\pi n^2}\right)\mathrm{D}. \qquad (4)$$

Identifying this formula with the formula of pneomenological electrodynamics $\mathrm{P}=[\epsilon-1]\ \mathrm{D}/4\pi\epsilon$, we get a "dispersion formula" for the longitudinal waves:

$$\frac{n^2}{\varepsilon(\omega)}=1-\sum_{j=1}^{s}a_{\parallel j}/(\Omega^2_{\parallel j}-\omega^2), \qquad (5)$$

$$a_{\parallel j}=4\pi n^2\alpha^2_{\parallel j}.$$

$\epsilon(\omega)$ should be the same in both cases, because the coefficient of proportionality between P and E should not depend on whether the fictitious dielectric polarization charges affect or do not affect the state of the field E. Identifying the expressions of n^2/ϵ from (5) and (3), we obtain the desired relation between the parameters $\Omega_{\parallel j}$, $\alpha_{\parallel j}$ and $\Omega_{\perp j}$.

In practice this is achieved most easily by considering the reciprocal of (3) as a rational fraction with respect to the variable ω^2 :

$$n^2/\varepsilon(\omega)=\prod(\omega^2)\bigg/\left[\prod(\omega^2)+\sum_{j=1}^{s}a_{\perp j}\prod{}^{'j}(\omega^2)\right]; \qquad (6)$$

where

$$\prod=\prod_{j=1}^{s}(\Omega^2_{\perp j}-\omega^2);\quad \prod{}^{'j}=(\Omega^2_{\perp 1}-\omega^2)\ldots$$
$$(\Omega^2_{\perp j-1}-\omega^2)(\Omega^2_{\perp j+1}-\omega^2)\ldots(\Omega^2_{\perp s}-\omega^2).$$

After having expressed this fraction in elementary fractions one has to identify each partial fraction with the corresponding term of the sum in the right hand side of (5).

A useful additional relation may be obtained by setting $\omega=0$ in (3):

$$\varepsilon_0\ \ n^2=1+\sum_{j=1}^{s}a_{\perp j}/\Omega^2_{\perp j}, \qquad (7)$$

where ϵ_0 is the static dielectric constant.

In the simplest case , when the number of ions in the elemenatry cell is two, $s=1$ and the expansion (6) in partial fractions has the form:

$$n^2/\varepsilon=1-a_\perp/(\Omega^2_\perp-\omega^2+a_\perp).$$

Identifying this expression with (5) we get $a_\parallel=a_\perp$ and $\Omega^2_\parallel=\Omega^2_\perp+a_\perp$ and it follows from (7) that $a_\perp=\Omega^2_\perp(\epsilon_0\ n^{-2}-1)$. Therefore:

$$\alpha_\parallel=\alpha_\perp/n^2,\qquad \Omega^2_\parallel=\Omega^2_\perp\ \varepsilon_0/n^2. \qquad (8)$$

In the case of a three ion cell, $s=2$, and the expansion (6) in elementary fractions and the identification of the result with (5) give

$$a_{\parallel 1}=[a_{\perp 1}(\Omega^2_{\perp 2}-\Omega^2_{\parallel 1}) \qquad (9)$$
$$+a_{\perp 2}(\Omega^2_{\perp 1}-\Omega^2_{\parallel 1})]/(\Omega^2_{\parallel 2}-\Omega^2_{\parallel 1})$$
$$=-(\Omega^2_{\perp 1}-\Omega^2_{\parallel 1})(\Omega^2_{\perp 2}-\Omega^2_{\parallel 1})/(\Omega^2_{\parallel 2}-\Omega^2_{\parallel 1}),$$
$$a_{\parallel 2}=-(\Omega^2_{\perp 1}-\Omega^2_{\parallel 2})(\Omega^2_{\perp 2}-\Omega^2_{\parallel 2})/(\Omega^2_{\parallel 1}-\Omega^2_{\parallel 2}),$$

where $\Omega^2_{\parallel 1}$ and $\Omega^2_{\parallel 2}$ are the roots of the quadratic equation

$$(\Omega^2_{\perp 1}-\omega^2)(\Omega^2_{\perp 2}-\omega^2)+a_{\perp 1}(\Omega^2_{\perp 2}-\omega^2)$$
$$+a_{\perp 2}(\Omega^2_{\perp 1}-\omega^2)=0,$$

where the unknown is ω^2 . The quantities $\alpha^2_{\perp j}$ and $\alpha^2_{\parallel j}$ can be expressed easily through $a_{\perp j}$ and $a_{\parallel j}$ using Eqs. (3) and (5).

In the case when the elementary cell consists of a large number of ions, one has to solve a higher order equation in ω^2. This is possible when the parameters of the transverse (longitudinal) vibrations are given numerically. The experimental determination of the parameters for the transverse vibrations can be made, for instance, by approximating through (3) the experimental dispersion curve for infrared light.

The relation between $\Omega_{\parallel}$ and $\Omega_{\perp}$ in the simplest particular case of two ions per cell, has been considered in earlier publications. It seems that Ref. 3 reports wrong relations. The relation (8) was derived for the first time in Ref. 4; it has later been derived by Tolpygo[1] and Callen[5] on the basis of the microtheory of ionic crystals; Huang[2] has derived it using the macroscopical electromagnetic theory.

In the case of an anisotropic crystal, if the orientation of one of its principal axes of polarization does not depend on the frequency (because of the symmetry of the crystal) the relation derived above between the parameters of longitudinal and transverse waves still hold for a longitudinal wave travelling along the said axis, and for a transverse wave travelling in a perpendicular direction and polarized along the principal axis. If the wave travels in an arbitrary direction it is, generally speaking, neither longitudinal nor transverse, and the relations derived above need a basic generalization.

The authors express their gratitude to L. D. Landau for his valuable remarks.

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Translated by E. S. Troubetzkoy
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Hyperfine Structure of the Paramagnetic Resonance Spectrum of $^{53}Cr^{3+}$ in Al_2O_3

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IN Ref. 1 the fine structure of the paramagnetic resonance spectrum of Cr^{3+} in monocrystalline Al_2O_3 was investigated. More detailed study of this spectrum showed that with concentrations of Cr $\sim 10^{-4}$ the fine structure lines, corresponding to various electronic transitions, have identical widths, equal to 13 G. Further decrease in the Cr concentration and also a lowering of the temperature of the crystal to the temperature of liquid nitrogen did not cause a narrowing of the lines. These data led to the conclusion, that the line width for small Cr concentrations depends mainly on the magnetic interaction of the spins of Cr^{3+} with the Al^{27} nuclei surrounding the Cr^{3+} ions in monocrystalline corundum.

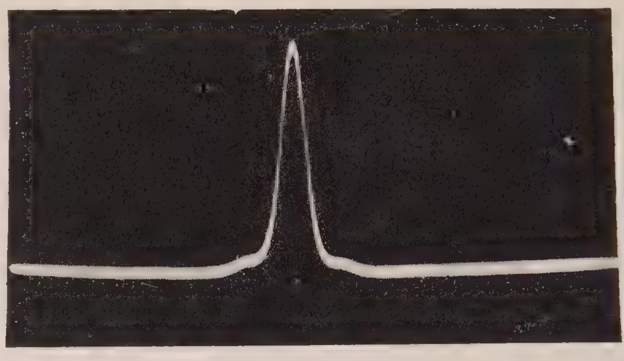

FIG. 1

For Cr concentrations greater than 10^{-4}, the line widths for various electronic transitions are not identical: the lines, corresponding to the transition $M = 3/2 \longleftrightarrow 1/2$ and $M = -1/2 \longleftrightarrow -3/2$ are wider than the line $M = 1/2 \longleftrightarrow -1/2$. This is connected with the fact that in this case the line width depends primarily on the magnetic interaction between the Cr^{3+} ions.

Relatively small line widths for small chromium concentrations permit one to observe hyperfine structure caused by the single stable odd isotope Cr^{53} whose concentration in the natural mixture of isotopes equals 9.4%. Figure 1 shows the oscillogram of the line corresponding to the electronic transition $M = 1/2 \longleftrightarrow -1/2$. Two components of the hyperfine structure caused by the Cr^{53} isotope may be clearly seen on the wings of the central line from the even chromium isotopes, whose nuclear spins are zero. The relative intensities of the central line and the two supplementary lines are in agreement with the values of the nuclear spin of Cr^{53}, $I = 3/2$[2], and the concentration of the Cr^{53} isotope in the natural mixture, 9.4%.

In order to resolve the hyperfine structure completely, we investigated the paramagnetic resonance in monocrystalline corundum, containing chromium,

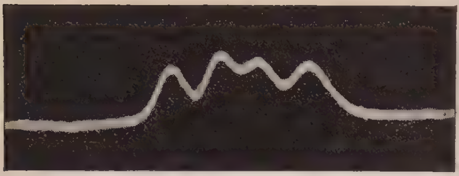

FIG. 2

enriched in the mass 53 isotope.

Figure 2 shows the oscillogram of the line $M = ½ \leftrightarrow -½$ in the case for a constant external magnetic field parallel to the axis of symmetry of the crystal. The hyperfine structure, consisting of the four lines, confirms the nuclear spin of Cr^{53} as equal to 3/2. The uneven spacing between the hyperfine components is caused by second order displacements. The hyperfine structure is described by the terms

$$AS_zI_z + B(S_xI_x + S_yI_y)$$

in the spin Hamiltonian, where S is the electron spin, I is the nuclear spin, A and B are hyperfine structure constants. From the experimental data we obtain the following values of the constants

$$|A| = (17.0 \pm 0.5) \cdot 10^{-4}\text{cm}^{-1}, \quad |B| \approx |A|.$$

Notice that the constant A of the hyperfine structure of Cr^{53} in corundum is near that observed in alums, but greater than that of cyanide.[2,3]

A detailed account of the results of the investigation of the fine and hyperfine structure of Cr^{3+} in corundum will be published later.

The authors wish to thank A. A. Popova for growing the single crystals, studied in the present work.

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66

Correlative Phenomena in *K*-Meson Capture

V. B. BERESTETSKII AND I. IA. POMERANCHUK

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THE process of capture of a K-meson by a proton, followed by a decay with emission of a hyperon, i.e., the reaction

$$K^- + p \rightarrow \Lambda^0 + \pi^0 \rightarrow p + \pi^- + \pi^0,$$

can be used to determine the spin of the Λ-particle from the angular correlation of the π-mesons.

If the spin of the K-meson is equal to zero, then the momentum of the initial system is equal to ½ (if the K-meson is captured in a S-state). The angular distribution $I_j(\theta)$ is given as a function of j and of the angle θ between the directions $\mathbf{n}_1$ and $\mathbf{n}_2$ of the relative momenta of the systems (Λ, π^0) and (p, π^-) $(I_{1/2}(\theta) = 1)$

$$\begin{aligned} I_{3/2}(\theta) &= 1 + P_2(\cos\theta) \sim 1 + 3\cos^2\theta; \\ I_{5/2}(\theta) &= 1 + {}^8/_7 P_2(\cos\theta) + {}^6/_7 P_4(\cos\theta) \\ &\sim 1 - 2\cos^2\theta + 5\cos^4\theta \end{aligned} \qquad (1)$$

(to be compared with the analogous formula for the decay of the Ξ-particle[1]). If the spin of the K-meson is 1, then the momentum of the initial system can be ½ as well as 3/2, and, therefore, the formula for the angular correlation are not unique.

If the system is in an external magnetic field, the dependence of the angular distribution on the field H can be used to determine the magnetic moment of the Λ-particle. In the presence of a magnetic field the correlation function has the form (see, e.g., Ref. 2):

$$\Omega = -kT \ln \sum_{n.N} \exp\left\{\frac{\mu N - E_{nN}}{kT}\right\}, \qquad (2)$$

where ω is the appropriate Larmor frequency, τ the lifetime of the Λ-particle and A_n the coefficient of P_n in Eq. (1). If the gyromagnetic ratio for the Λ-particle is equal to that of the proton, then $\omega\tau$ reaches the value ≈ 0.3 for $H = 3 \times 10^4$ G.

For the $j = 3/2$, Eq. (2) takes the form

$$\begin{aligned} I = 1 &+ P_2(\cos\theta_1) P_2(\cos\theta_2) \\ &+ {}^3/_4 \sin 2\theta_1 \sin 2\theta_2 [\cos(\varphi_1 - \varphi_2) \\ &- \omega\tau \sin(\varphi_1 - \varphi_2)] / (1 + \omega^2\tau^2) \\ &+ {}^3/_4 \sin^2\theta_1 \sin^2\theta_2 [\cos 2(\varphi_1 - \varphi_2) \\ &- 2\omega\tau \sin 2(\varphi_1 - \varphi_2)] / (1 + 4\omega^2\tau^2), \end{aligned} \qquad (3)$$

where $\theta_1, \varphi_1, \theta_2, \varphi_2$ are the angles of the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ in a system of spherical coordinates, when the z-axis is along the magnetic field.

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Nonlinear Theory of Longitudinal Plasma Oscillations

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IN Refs. 1 and 2 there was considered the one dimensional oscillations of the electronic plasma, without taking into account the temperature effects, on the assumption that the electron density n, the electrical field E and the electron speed v are linear functions of the combination $x - Vt$, where V is the velocity of the wave, propagating along the x-axis. In the present note we obtain a fundamental result of this work —a relation between the frequency of the longitudinal wave and the amplitude for a more general assumption, namely, for an arbitrary relation between the quantities n, E and v and x and t. The entire derivation is very much simplified if the basic equations are transformed from the Eulerian to the Lagrangian form.

We write the equation of motion of the electron and Maxwell's equations

$$dp/dt = eE, \tag{1}$$

$$dE/\partial x = 4\pi e\,(n - n_0), \tag{2}$$

$$0 = (1/c)\,(\partial E/dt) + (4\pi/c)\,env, \tag{3}$$

where n_0 is the ion density, which is assumed to be fixed; e, m and p are the charge, mass and momentum of the electron. From (2) and (3) follow

$$dE/dt = \partial E/\partial t + v\partial E/\partial x = -4\pi e n_0 v. \tag{4}$$

Differentiating Eq. (1) with respect to time and comparing with (4), we obtain

$$d^2p/dt^2 + \omega_0^2 mv = 0, \quad \omega_0^2 = 4\pi^2 n_0/m. \tag{5}$$

In the non-relativistic case $p = mv$ and, consequently $(d^2 v/dt^2) + \omega_0^2 v = 0$, i.e., the frequency of the plasma oscillation does not depend on the amplitude.[1]

In the relativistic case we express the velocity in terms of the momentum,

$$\frac{d^2p}{dt^2} + \frac{\omega_0^2}{\sqrt{1 + p^2/m^2c^2}}\, p = 0,$$

from which the dependence of the frequency on amplitude follows directly, as was obtained in Ref. 2.

In conclusion the author wishes to thank Ia. B. Fainberg for valuable suggestions and Prof. A. I. Akhiezer and G. Ia. Luibarskii for discussions.

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72

A Note on Mixed Meson Theory

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FEYNMAN[1] has made the statement that in a mixed theory of scalar and vector mesons with vector coupling, nonrenormalizable infinities cancel if the coupling constants are equal. Reference 2 is devoted to the application of this theory. Such a statement also occurs in Ref. 3. Actually, however, the cancellation of nonrenormalizable infinities is equivalent in this case to the fact that the equation for the vector meson

$$(p^2\delta_{\nu\mu} - p_\nu p_\mu)\,\varphi_\mu = -s_\nu$$

is transformed into

$$p^2\varphi_\nu = -s_\nu.$$

In this case the field φ_ν describes particles with spin one and zero, where components with spin zero correspond to a negative energy (see Ref. 4). This circumstance is also noted in Ref. 1.

Starting from a Hermitian Lagrangian for two fields with spin zero and one, interacting with the nuclear field, one finds that the nonrenormalizable divergences do not cancel, but, on the contrary, are additive. For each vector meson line (included in the vertex) one must write

$$f_1^2(\gamma_\nu \cdots \gamma_\mu)\ (\delta_{\nu\mu} - k_\nu k_\mu \mu_1^{-2})/(k^2 - \mu_1^2), \qquad (1)$$

where the dots denote an arbitrary part of the diagram, and for each scalar meson line, one must write

$$-(f_2^2/\mu_2^2)\,(\mathrm{k}\ \ldots\ \mathrm{k})\,(k^2 - \mu_2^2)^{-1}. \qquad (2)$$

The minus sign in the last expression is due to the fact that one of the factors k describes the creation of a meson, while the other describes annihilation (this sign also follows from Feynman's rule[1]: k is the difference between the initial and final momentum of the nucleon).

It follows from (1) and (2) that the divergences connected with the factor k in the numerator, will only cancel if $f_1^2/\mu_1^2 = -f_2^2/\mu_2^2$, i.e., if one of the charges is imaginary.

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Relative Cross Sections for $n-p$ Reactions Involving Nuclei with Several Stable Isotopes

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(August, 1956)

WHILE examining the literature on nuclear reactions the author noticed a regular variance in the $n-p$ and $n-\alpha$ reaction cross sections (for 14 mev neutrons) in the stable isotope series of individual elements. It follows from the data of Clarke and others[1] that the $n-p$ and $n-\alpha$ reaction cross sections for the various isotopes of an element decrease with increasing isotopic mass number as a rule (in seven of the nine investigated cases); moreover, in six of the seven cases, they decrease almost exactly by a factor of two or four*. In order to verify this regularity and define it more precisely, a program was initiated for the experimental determination of $n-p$ reaction cross sections of nuclei with several stable isotopes (Zr, Cd, Ti, Sr and Ca).

Specimens of salts of the elements under investigation were bombarded with 14 mev neutrons and dissolved; radioactive isotopes of Y, Ag, Sc, Rb and K produced by the $n-p$ reactions were then separated from the solution and their activity was measured with a standard cylindrical geiger counter. An analysis of the decay curves then yielded the activity of each isotope from the time the bombardment ended. Absolute activities were computed by correcting for decay during bombardment time and for absorption of radiation by the walls of the counter. The latter correction was determined in each case by means of special absorption measurements. This correction is not very large inasmuch as the reaction products of all investigated reactions [except for the reaction Ti(n, p) Sc] emit hard β-rays; therefore, even a large error in the determination of this correction cannot seriously affect the final result.

In this fashion relative cross sections were obtained for four isotopes of Zr, four of Cd, two of Sr, two of Ca; only rough preliminary results were obtained for Ti:

$$\sigma\,\mathrm{Zr}^{90} : \sigma\,\mathrm{Zr}^{91} : \sigma\,\mathrm{Zr}^{92} : \mathrm{Zr}^{94} = 1 : 0.74 : 0.46 : 0.20;$$

$$\sigma\,\mathrm{Cd}^{106} : \sigma\,\mathrm{Cd}^{111} : \sigma\,\mathrm{Cd}^{112} : \sigma\,\mathrm{Cd}^{113} = 5{,}00 : 1 : 0.71 : 0.52;$$

$$\sigma\,\mathrm{Sr}^{86} : \sigma\,\mathrm{Sr}^{88} = 1 : 0.46;$$

$$\sigma\,\mathrm{Ca}^{42} : \sigma\,\mathrm{Ca}^{44} = 1 : 0{,}24;$$

$$\sigma\,\mathrm{Ti}^{47} : \sigma\,\mathrm{Ti}^{48} : \sigma\,\mathrm{Ti}^{49} = (1) : (0.25) : (0{,}06)$$

Similar relations are presented below for five pairs of other isotopes computed from Clarke's data.

$$\sigma\,\mathrm{Mg}^{24} : \sigma\,\mathrm{Mg}^{25} = 1 : 0.23;\quad \sigma\,\mathrm{Si}^{28} : \sigma\,\mathrm{Si}^{29} = 1 : 0.46;$$

$$\sigma\,\mathrm{S}^{32} : \sigma\,\mathrm{S}^{34} = 1 : 0.23;\quad \sigma\,\mathrm{Zn}^{64} : \sigma\,\mathrm{Zn}^{66} = 1 : 0.26;$$

$$\sigma\,\mathrm{Ge}^{70} : \sigma\,\mathrm{Ge}^{72} = 1 : 0{,}50.$$

The following deductions can be made from the adduced data:

1) the $n-p$ reaction cross section for nuclei with several isotopes decreases considerably as

a rule, with increasing isotopic mass. This decrease is more noticeable for light nuclei and less so for heavy ones;

2) for the case of light nuclei, increasing the mass number by two, results in a 4-fold (Ca, Si, S, Zn) or a 16-fold [Mg, (Ti)] decrease in the reaction cross section, while a 2-fold decrease occurs for the heavy elements (Sr, Zr, Cd, Ge);

3) the results which have been obtained apparently disagree with the statistical theory of nuclear reactions, and perhaps provide evidence for the preservation of a shell structure even at high excitation energies.

Some of the data cited above were obtained by O. I. Ivanov and E. E. Vinogradov', students at Moscow State University, in fulfillment of their theses.

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* There is evidence to suggest that Clarke's data, which represents an exception to this rule ($Zr^{90,94}$, $Pd^{104,105}$, $Ge^{72,73}$), is uncertain because the experimental method is less reliable in these cases.

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